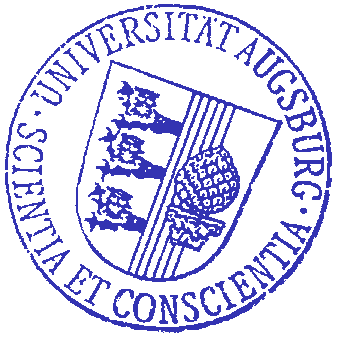
Relativistic Brownian Motion and Diffusion Processes

Dissertation

Institut für Physik

Mathematisch-Naturwissenschaftliche Fakultät

Universität Augsburg



eingereicht von

Jörn Dunkel

Augsburg, Mai 2008

Erster Gutachter:	Prof. Dr. Peter Hänggi
Zweiter Gutachter:	Prof. Dr. Thilo Kopp
Dritter Gutachter:	Prof. Dr. Werner Ebeling

Tag der mündlichen Prüfung: 22.07.2008

Contents

Symbols

M	rest mass of the Brownian particle
m	rest mass of a heat bath particle
Σ	inertial laboratory frame := rest frame of the heat bath
Σ'; Σ_*	moving frame; comoving rest frame of the Brownian particle
$\mathcal{O}$; $\mathcal{O}'$	lab observer; moving observer
t	time coordinate
τ	proper time of the Brownian particle
c	vacuum speed of light (set to unity throughout, i.e., $c = 1$)
d	number of space dimensions
$\boldsymbol{X}, \boldsymbol{x}$	position coordinate
$\boldsymbol{V}, \boldsymbol{v}$	particle velocity
$\boldsymbol{w}$	observer velocity
$\boldsymbol{P}, \boldsymbol{p}$	momentum coordinates
E, ϵ	particle energy
$\eta = (\eta_{\alpha\beta})$	Minkowski metric tensor
Λ	Lorentz transformation (matrix)
γ	Lorentz factor. $\gamma(\boldsymbol{v}) = (1 - \boldsymbol{v}^2)^{-1/2}$
X^α	(contravariant) time-space four-vector $(X^\alpha) = (t, \boldsymbol{X})$, $\alpha = 0, 1, \ldots, d$
P^α	energy-momentum four-vector, $(P^\alpha) = (E, \boldsymbol{P})$
U^α	velocity four-vector, $U^\alpha = P^\alpha / M$
f	one-particle phase space probability density
ϱ	one-particle position probability density
ϕ	one-particle momentum probability density
ψ	one-particle velocity probability density
k_{B}	Boltzmann constant (set to unity throughout, i.e., $k_{\mathrm{B}} = 1$)
$\mathcal{T}$	temperature
β	inverse thermal energy $\beta := (k_{\mathrm{B}}\mathcal{T})^{-1}$
$\mathfrak{S}$	relative entropy
α	friction coefficient

D	noise amplitude
$\mathcal{D}$	spatial diffusion constant
$\boldsymbol{B}(s)$	d-dimensional standard Wiener process with time parameter s
$\mathbb{P}$	probability measure of the Wiener process
$*$	Ito (pre-point) interpretation of the stochastic integral
$\circ$	Stratonovich-Fisk (mid-point) interpretation of the stochastic integral
$\bullet$	backward Ito (post point) interpretation of the stochastic integral
$\mathbb{N}$	set of natural numbers $1, 2, \ldots$
$\mathbb{Z}$	set of integer numbers
$\mathbb{R}$	set of real numbers
λ	Lebesgue measure
μ, ρ	measures
$\langle X \rangle$	expected value of a random variable X

Chapter 1

Introduction and historical overview

In his *annus mirabilis* 1905 Albert Einstein published four manuscripts [1–4] that would forever change the world of physics. Two of those papers [2, 3] laid the foundations for the special theory of relativity, while another one [4] solved the longstanding problem of classical (nonrelativistic) Brownian motion.[1] Barring gravitational effects [5, 6], special relativity has proven to be the correct framework for describing physical processes on all terrestrial scales [7, 8]. Accordingly, during the past century extensive efforts have been made to adapt established nonrelativistic theories such as, e.g., thermodynamics, quantum mechanics or field theories [9] to the requirements of special relativity. Following this tradition, the present thesis investigates how stochastic concepts such as Brownian motion may be generalized within the framework of special relativity. The subsequent chapters intend to provide a cohesive summary of results obtained during the past three years [10–17], also taking into account important recent contributions by other authors (see, e.g., [18–24]).

Historically, the term 'Brownian motion' refers to the irregular dynamics exhibited by a test particle (e.g., dust or pollen) in a liquid environment. This phenomenon, already mentioned by Ingen-Housz [25, 26] in 1784, was first analyzed in detail by the Scottish botanist Robert Brown [27] in 1827. About 80 years later, Einstein [4], Sutherland [28] and von Smoluchowski [29] were able to theoretically explain these observations. They proposed that Brownian motion is caused by quasi-random, microscopic interactions with molecules forming the liquid. In 1909 their theory was confirmed experimentally by Perrin [30], providing additional evidence for the atomistic structure of matter. During the first half of the 20th century the probabilistic description of Brownian motion processes was further elaborated in seminal papers by Langevin [31, 32], Fokker [33], Planck [34], Klein [35], Uhlenbeck and Ornstein [36] and Kramers [37]. Excellent reviews of these early contributions are given by Chandrasekhar [38] and Wang and Uhlenbeck [39].

[1] Einstein's first paper [1] provided the theoretical explanation for the photoelectric effect.

In parallel with the studies in the field of physics, outstanding mathematicians like Bachelier [40], Wiener [41–43], Kolmogoroff [44–46], Feller [47], and Lévy [48, 49] provided a rigorous basis for the theory of Brownian motions and stochastic processes in general. Between 1944 and 1968 their groundbreaking work was complemented by Ito [50,51], Gihman [52–54], Fisk [55,56] and Stratonovich [57–59], who introduced and characterized different types of stochastic integrals or, equivalently, stochastic differential equations (SDEs). The theoretical analysis of random processes was further developed over the past decades, and the most essential results are discussed in several excellent textbook references [60–66][2]. The modern theory of stochastic processes goes far beyond the original problem considered by Einstein and his contemporaries, and the applications cover a wide range of different areas including physics [67–74], biology [75, 76], economy and finance [77–79].

The present thesis is dedicated to the question how SDE-based Brownian motion models can be generalized within the framework of special relativity. In the physics literature [65], SDEs are often referred to as Langevin equations [31, 32], and we shall use both terms synonymously here. From a mathematical point of view, SDEs [64] determine well-defined models of stochastic processes; from a physicist's point of view, their usefulness for the description of a real system is *a priori* an open issue. Therefore, the derivation of nonrelativistic Langevin equations from microscopic models has attracted considerable interest over the past 60 years [13,80–86]. Efforts in this direction helped to clarify the applicability of SDEs to physical problems and led, among others, to the concept of quantum Brownian motion [82, 87–99].[3]

If one aims at generalizing the classical Brownian motion concepts to special relativity, then several elements from relativistic equilibrium thermodynamics and relativistic statistical mechanics play an important role. The first papers on relativistic thermodynamics were written by Einstein [109] and Planck [110, 111] in 1907. A main objective of their studies was to clarify the Lorentz transformation laws of thermodynamic variables (temperature, pressure, etc.).[4] In 1963 the results of Einstein and Planck were questioned by Ott [115], whose work initiated an intense debate about the correct relativistic transformation behavior of thermodynamic quantities [116–160].[5] However, as clarified by van Kampen [137] and Yuen [161], the controversy surrounding relativistic thermodynamics can be resolved by realizing that thermodynamic quantities can be *defined* in different,

[2]The history of the mathematical literature on Brownian motions and stochastic processes is discussed extensively in Section 2.11 of Ref. [64]; see also Chapters 2-4 in Nelson [61].

[3]The vast literature on classical Brownian motion processes and their various applications in nonrelativistic physics is discussed in several survey articles [68–73, 100–104]. Nonrelativistic generalizations of the standard theory as, e.g., anomalous diffusion processes have been summarized in [73, 105, 106], while review articles on nonrelativistic quantum Brownian motion can be found in [97, 98, 107, 108].

[4]See also Pauli [112], Tolman [113] and van Dantzig [114] for early discussions of this problem.

[5]The pre-1970 literature on this disputed issue has been reviewed by Yuen [161] and Ter Haar and Wegland [162]; more recent surveys can be found in [163–165].

equally consistent ways.

While some authors considered relativistic thermodynamics as a purely macroscopic theory, others tried to adopt a more fundamental approach by focussing on relativistic equilibrium statistical mechanics. Pioneering work in the latter direction is due to von Mosengeil [166], who studied the radiation of a moving cavity, and Jüttner [167], who derived in 1911 the relativistic generalization of Maxwell's velocity distribution [168].[6] Two decades later, in 1928, Jüttner [170] also calculated the equilibrium distributions for ideal relativistic quantum gases. Relativistic generalizations of equipartition and virial theorems [171] were discussed by Pauli [112] in 1921 and Einbinder [172] in 1948.[7] Research on relativistic equilibrium thermostatistics experienced its most intense phase between 1950 and 1970 [124, 125, 133, 137, 146, 176–191]. An excellent exposition on the conceptual foundations and difficulties of relativistic statistical mechanics was given by Hakim [192–194] in 1967. During the past years the field has continued to attract interest [14,17,173,195–219].[8]

The recurring debate on relativistic thermostatistics can be traced back to the difficulty of treating many-particle interactions in a relativistically consistent manner. In nonrelativistic physics interactions may propagate at infinite speed, i.e., they can be modelled via instantaneous interaction potentials which enter additively in the Hamilton function; from that point on, nonrelativistic statistical mechanics emerges without much difficulty [220, 221]. Unfortunately, the situation becomes significantly more complicated in the relativistic case: Due to their finite propagation speed, relativistic interactions should be modelled by means of fields that can exchange energy with the particles [6]. These fields add an infinite number of degrees of freedom to the particle system. Elimination of the field variables from the dynamical equations may be possible in some cases but this procedure typically leads to retardation effects, i.e., the particles' equations of motions become non-local in time [192, 193, 222–225]. Thus, in special relativity it is usually very difficult or even impossible[9] to develop a consistent field-free Hamilton formalism for interacting many-particle systems [225–228].

Seminal contributions to the theory of relativistic many-particle interactions were provided by Fokker [229], Wheeler and Feynman [222, 223], Pryce [230], Havas and Goldberg [231, 232], and Van Dam and Wigner [224, 225]. Over the past decades several no-interaction theorems were proven [225–228] that, within their respective qualifications, forbid certain types of interaction models within the framework of special relativity. The mathematical structure of relativistic many-particle interactions was analyzed in detail by

[6] See also Synge's textbook classic [169].

[7] More recent investigations related to these specific topics can be found in [173–175].

[8] In particular, some recent papers [14, 206, 209, 211, 215–217] have raised doubts about the correctness of Jüttner's equilibrium distribution [167, 170], but relativistic molecular dynamics simulations confirm Jüttner's prediction [17, 214]; cf. Section 3.2 below.

[9] An exception is the one-dimensional gas of point particles with strictly localized elastic interactions (cf. discussion in Chapter 3 below).

Arens and Babbitt [233], and various semi-relativistic approximations have been discussed, e.g., in [234–236].[10] Another, intensely studied method for describing relativistic interactions is based on the so-called constraint formalism [200, 211, 248–264]. The foundations of this approach were worked out by Dirac [248] in 1949, who aimed at constructing a consistent relativistic quantum theory. However, compared with the nonrelativistic case, it seems fair to say that neither of the various formulations has led to a relativistic statistical many-particle theory that is on the same rigorous, commonly accepted footing as its nonrelativistic counterpart.[11]

In spite of the difficulties impeding a rigorous treatment of relativistic many-particle systems, considerable progress has been made over the past century in constructing an approximate relativistic kinetic theory based on *one-particle* phase space probability density functions (PDFs). Early pioneering work that paved the way for the relativistic generalization of the nonrelativistic Boltzmann equation [267,268] was done by Eckart [269], Lichnérowicz and Marrot [270], Kluitenberg et al. [271], Beliaev and Budker [272], Synge [169], and Israel [273].[12] Comprehensive introductions to relativistic kinetic theory can be found in the textbooks by Stewart [288], de Groot et al. [289], and Cercignani and Kremer [290], or also in the reviews by Ehlers [291] and Andréasson [292].[13]

From relativistic kinetic theory [289, 290] it is only a relatively small step to formulating a theory of relativistic Brownian motion processes in terms of Fokker-Planck equations (FPEs) and Langevin equations. While the relativistic Boltzmann equation is a nonlinear partial integro-differential equation, FPEs are linear partial differential equations and, therefore, can be solved more easily [63]. In this work, we will mostly focus on relativistic stochastic processes that are characterized by linear evolution equations for their respective one-particle (transition) PDFs. The research in this direction may be roughly divided into four different areas, although, of course, there are substantial overlaps and intersections between them:

a) *Relativistic Fokker-Planck equations in phase space.* Similar to the relativistic Boltzmann equation, relativistic FPEs can be used to model non-equilibrium and relaxation phenomena in relativistic many-particle systems. Generally, an FPE can be derived from a Langevin equation or as an approximation to a more general linear master equation governing the stochastic process [65, 293]. Yet another way of de-

[10] Kerner [237] has edited a reprint collection covering large parts of the pre-1972 literature on relativistic action-at-a-distance models, and more recent contributions can be found in [199, 200, 238–247].

[11] For a more detailed discussion of relativistic many-particle theory, we refer to the insightful considerations in the orginal papers of Van Dam and Wigner [224, 225] and Hakim [192–194, 265] as well as to the recent review by Hakim and Sivak [266].

[12] See also [138, 153, 196, 234, 274–287].

[13] Although standard relativistic kinetic theory can be considered as well-established nowadays [289,290], some authors questioned its validity in recent years and proposed modifications of the relativistic Boltzmann equation [206,215–217]. Recent numerical simulations [17,214] support the standard theory [289,290].

riving an FPE is to approximate the collision integrals in the nonlinear Boltzmann equation by a differential expression that contains effective friction and diffusion coefficients [278,294]. The latter approach has been applied successfully in different areas of physics over the past decades, including plasma physics [278, 295–307], high energy physics [308–314], and astrophysics [315–320]. For example, in 1970 Akama [278] started from the relativistic Boltzmann equation to construct the FPE for a relativistic plasma. In the 1980/1990s this approach was further elaborated [295–303] and several numerical methods for solving FPEs were developed [298, 321–323]. During the past three decades stochastic concepts assumed an increasing importance in other areas of high energy physics as well. In the early 1980s relativistic Fokker-Planck-type equations played an important role in the debate about whether or not the black body radiation spectrum is compatible with Jüttner's relativistic equilibrium distribution [173, 324–327]. More recently, FPEs have also been used to model diffusion and thermalization processes in quark-gluon plasmas, as produced in relativistic heavy ion collision experiments [308–310, 312–314]. Similarly, the combination of probabilistic and relativistic concepts can be useful to describe complex high energy processes in astrophysics [315–320, 328].

b) *Relativistic Langevin equations.* A complementary approach towards relativistic stochastic processes is based on relativistic Langevin equations [10,11,13,18–22,329–335]. The latter yield explicit sample trajectories for the stochastic motion of a relativistic Brownian particle. They are, therefore, particularly useful for numerical simulations. Relativistic Langevin equations may either be postulated as phenomenological model equations [10, 18] or derived from more precise microscopic models by imposing a sequence of approximations [13]. Compared with the nonrelativistic case, the latter task becomes considerably more complicated due to the aforementioned difficulties in classical relativistic many-particle theory. The phenomenological Langevin approach to relativistic Brownian motion was developed by Debbasch et al. [18], who in 1997 proposed a simple relativistic generalization of the classical Ornstein-Uhlenbeck process [36], called ROUP. As will be discussed in Chapter 4, the ROUP may be considered as a special limit cases of a larger class of relativistic Langevin processes [12]. Furthermore, complementing the phenomenological Langevin theory of relativistic Brownian motions, we will analyze the assumptions and approximations that must be made in order to obtain a relativistic Langevin equations from a 1D microscopic binary collision model. From a practical point of view, relativistic Langevin equations provide a useful tool for modelling the dynamics of relativistic particles in a random environment, since these SDEs may be simulated by using well-established Monte-Carlo techniques that are numerically robust and efficient [66, 79, 336]. Very recently, relativistic Langevin equations have been employed by van Hees et al. [312, 313] and

Rapp et al. [314, 337], who analyzed thermalization effects in quark-gluon plasmas, and also by Dieckmann et al. [338], who studied the thermalization in ultrarelativistic plasma beam collisions as common in astrophysical settings.

c) *Mathematically oriented research.* To our knowledge, the first detailed mathematical studies on relativistic diffusion processes were performed independently by Łopuszańnski [339] in 1953, Rudberg [340] in 1957, and Schay [293] in 1961. The work of these authors was further elaborated by Dudley who published between 1965 and 1974 a series of papers [341–344] that aimed at providing an axiomatic approach to Lorentz invariant Markov processes in *phase space*. Independently, a similar program was pursued by Hakim [192–194, 345, 346] between 1965 and 1968. Hakim not only derived different forms of relativistic FPEs [345, 346], his insightful analysis also elucidated the conceptual subtleties of relativistic stochastic processes [346] and relativistic statistical mechanics [192, 193, 265]. Dudley (Theorem 11.3 in [341]) and Hakim (Proposition 2 in [346]) proved the non-existence of nontrivial[14] Lorentz invariant Markov processes in Minkowski space, as already suggested in Łopuszańnski's early work [339]. This important result implies that it is nontrivial to find acceptable relativistic generalizations of the well-known nonrelativistic diffusion equation [220]

$$\frac{\partial}{\partial t}\varrho(t,x) = \mathcal{D}\,\nabla^2\varrho(t,x). \tag{1.1}$$

Put differently, if one wishes to model relativistic random motions by means of a Markov process [64] with respect to coordinate time t then phase space coordinates have to be used (i.e., position *and* momentum). The mathematical interest in relativistic diffusion processes increased in the 1980s and 1990s, when several authors considered the possibility of extending Nelson's stochastic quantization approach [347] to the framework of special relativity, see e.g. [348–369] and also Section III.H in [370].[15] Important recent results on classical relativistic diffusions are due to Angst and Franchi [23], who were able to characterize the asymptotic behavior of a large class of special relativistic Brownian motion processes on phase space by means of a Central Limit Theorem.[16]

[14] A diffusion process is considered as 'nontrivial' if a typical path has a non-constant, non-vanishing velocity.

[15] These studies, although interesting from a mathematical point of view, appear to have little physical relevance because Nelson's stochastic dynamics [347] fails to reproduce the correct quantum correlation functions even in the nonrelativistic case [371]. Therefore, the present work focusses primarily on relativistic *non*-quantum diffusion processes.

[16] In this context, we also mention the recent work by Rapoport [372, 373] and Franchi and Le Jan [374], who extended the approach of Dudley [341–344] to the framework of general relativity.

d) *Non-Markovian generalizations of the nonrelativistic diffusion equation* (1.1). A commonly considered 'relativistic' generalization of Eq. (1.1) is the telegraph equation [375–377]

$$\tau_v \frac{\partial^2}{\partial t^2} \varrho(t,x) + \frac{\partial}{\partial t} \varrho(t,x) = \mathcal{D}\, \nabla^2 \varrho(t,x), \tag{1.2}$$

with $\tau_v > 0$ denoting a finite relaxation time scale. Unlike the classical diffusion equation (1.1), which is recovered for $\tau_v = 0$, the telegraph equation (1.2) contains a second order time-derivative and, therefore, describes a non-Markovian process. While the classical diffusion equation (1.1) permits superluminal propagation speeds, the diffusion fronts described by Eq. (1.2) travel at finite absolute velocity $v = (\mathcal{D}/\tau_v)^{1/2}$. Masoliver and Weiss [377] discuss four possibilities of deriving Eq. (1.2) from different underlying models. The first probabilistic derivation of Eq. (1.2) for the 1D case was given by Goldstein [375] in 1950. His approach was based on a so-called persistent random walk model originally introduced by Fürth [378, 379] in 1917 as a paradigm for diffusive motion in biological systems and later also considered by Taylor [380] in an attempt to treat turbulent diffusion.[17] In contrast to standard non-directed random walk models, which lead to the classical diffusion equation (1.1) when performing an appropriate continuum limit [64], the random jumps of a persistent walk take into account the history of a path by assigning a larger probability to those jumps that point in the direction of the motion before the jump [375, 376]. Persistent random walk models can be used to describe the transmission of light in multiple scattering media [383] such as foams [384–386] and thin slabs [382, 387]. Similarly, the telegraph equation (1.2) has been applied in various areas of physics over the past decades, e.g., to model the propagation of electric signals and heat waves.[18] An interesting connection between the free particle Dirac equation [397] and the telegraph equation (1.2) was pointed out by Gaveau et al. [398] in 1984: The solutions of both equations may be linked by means of an analytic continuation quite similar to the relation between the diffusion equation (1.1) and the free particle Schrödinger equation in the nonrelativistic case.[19] On the other hand, the telegraph equation (1.2) is not the only possible generalization of Eq. (1.1) and a rather critical discussion of Eq. (1.2) in the context of relativistic heat transport was given by van Kampen [140] in 1970. In Chapter 5 we will take a closer look at the properties of Eq. (1.2) and address potential alternatives [15, 405].

[17] See also Kac [376] and Boguñá et al. [381, 382].

[18] A detailed review of the pre-1990 research on heat waves was provided by Joseph and Preziosi [388, 389], while more recent discussions and applications of Eq. (1.2) can be found in [377, 390–396].

[19] For further reading about path integral representations of the Dirac propagator we refer to the papers of Ichinose [399, 400], Jacobson and Schulman [401], Barut and Duru [402], and Gaveau and Schulman [403]; see also footnote 7 in Gaveau et al. [398] and problem 2-6, pp. 34-36 in Feynman and Hibbs [404].

Concluding this brief historical overview, we may summarize that the theory of relativistic Brownian motion and diffusion processes has experienced considerable progress during the past decade, with applications in various areas of high energy physics [308, 310–312, 314, 393, 394, 406] and astrophysics [318, 320, 328, 338]. From a general perspective, relativistic stochastic processes provide a useful approach whenever one has to model the quasi-random behavior of relativistic particles in a complex environment. Therefore, it may be expected that relativistic Brownian motion and diffusion concepts will play an increasingly important role in future investigations of, e.g., thermalization and relaxation processes in astrophysics or high energy collision experiments. The present work aims to provide a comprehensive overview of the theory of relativistic Brownian motions with a particular emphasis on relativistic Langevin equations. For this purpose, the subsequent parts are organized as follows. Chapter 2 summarizes the Langevin theory of nonrelativistic Brownian motions in phase space. Chapter 3 discusses relevant aspects of relativistic equilibrium thermostatistics. Relativistic Langevin equations and their associated FPEs, are considered in Chapter 4. Chapter 5 is dedicated to relativistic diffusion processes in Minkowski space-time; as outlined above, such processes must necessarily be non-Markovian. The thesis concludes with a summary of open questions in Chapter 6, which may serve as a starting point for future investigations and extensions of the theory. In order to present the most important ideas and concepts in a transparent way, the discussion in the main text will focus mostly on the simplest case of one space dimension (1D). The generalization to higher space dimensions is usually straightforward and the corresponding equations are summarized in the Appendix D.

Chapter 2

Nonrelativistic Brownian motion

In order to briefly introduce the underlying mathematical concepts, we first recall some basic definitions and results from the Langevin theory [31, 32] of nonrelativistic Brownian motions. The Langevin and Fokker-Planck equations discussed in this part will be useful later on, because they represent the nonrelativistic limit case of the relativistic theory, which will be developed in Sections 4 and 5. The condensed discussion of nonrelativistic Brownian motion processes in Section 2.1 is primarily based on the papers of Uhlenbeck and Ornstein [36], Wang and Uhlenbeck [39], and Klimontovich [101]. For further reading about nonrelativistic stochastic processes and their numerous applications in physics and mathematics, we refer to the review articles of Chandrasekhar [38], Fox [100], Hänggi and Thomas [67], Bouchaud and Georges [105], Metzler and Klafter [106], Hänggi and Marchesoni [71], Frey and Kroy [72], or the textbooks [64–66, 407].

The present chapter is structured as follows. We begin by discussing the linear Langevin equation of the classical nonrelativistic Ornstein-Uhlenbeck process. Subsequently, nonlinear generalizations of this process will be considered. In this context, we will address the choice of discretization rules and generalized fluctuation-dissipation theorems. These issues will become important again later on, when we discuss the Langevin theory of relativistic Brownian motions in Chapter 4. The last part of this chapter focusses on the question how stochastic differential equations (SDEs) can be derived from microscopic models. As typical examples, the well-known harmonic oscillator model [80–86] and a recently proposed binary collision model [13] will be considered. In contrast to the oscillator model, the collision model can be generalized to the framework of special relativity, and its relativistic version will be discussed in Section 4.3.

2.1 Langevin and Fokker-Planck equations

2.1.1 Linear Brownian motion: Ornstein-Uhlenbeck process

We consider a point-like Brownian particle (mass M) surrounded by a stationary homogeneous heat bath consisting, e.g., of smaller liquid particles (mass $m \ll M$) at constant temperature $\mathcal{T}$. The inertial rest frame[1] Σ of the heat bath will be referred to as *lab frame* hereafter. The position of the Brownian particle in Σ at time t is denoted by $X(t)$ and its velocity is given by $V(t) := \mathrm{d}X(t)/\mathrm{d}t$. The associated nonrelativistic momentum of the Brownian particle is defined by $P(t) := MV(t)$.

Free Brownian motion The standard paradigm for a free nonrelativistic Brownian motion process in the absence of external forces is the Ornstein-Uhlenbeck process. The Ornstein-Uhlenbeck process is determined by the Langevin equations [31, 32, 36, 38, 39]

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \frac{P}{M}, \tag{2.1a}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -\alpha P + (2D)^{1/2} * \zeta(t). \tag{2.1b}$$

The first term on the rhs. of Eq. (2.1b) is the linear friction force, where the constant friction coefficient $\alpha > 0$ represents an inverse relaxation time. The stochastic Langevin force $L(t) = (2D)^{1/2} * \zeta(t)$ models the fluctuations in the heat bath.[2] In the case of the Ornstein-Uhlenbeck process, the amplitude of these fluctuations is tuned by the constant noise parameter $D > 0$, and the Gaussian white noise process $\zeta(t)$ is characterized by:

$$\langle \zeta(t) \rangle = 0, \tag{2.2a}$$

$$\langle \zeta(t)\, \zeta(s) \rangle = \delta(t-s), \tag{2.2b}$$

with all higher cumulants being zero. In Eqs. (2.2), $\langle \cdot \rangle$ is understood as an average over all possible realizations of the noise process $\zeta(s)$. We summarize the physical assumptions, implicitly underlying Eqs. (2.1) and (2.2):

- The heat bath is spatially homogeneous and stationary; i.e., relaxation processes within the heat bath occur on time scales much shorter than the relevant dynamical time scales associated with the motion of the heavy Brownian particle.

[1] By definition, the mean velocity of the heat bath particles vanishes in Σ.

[2] Throughout, the symbol '$*$' is used to denote Ito's stochastic integral definition. A precise specification of the employed stochastic integral convention (i.e., discretization rule) becomes relevant, if one wishes to consider a momentum dependent noise amplitude $D(P)$ and/or nonlinear transfomations of the momentum process $P(t)$. The most commonly used stochastic integral definitions and their implications are discussed in App. C. However, for our present discussion in Section 2.1.1, it suffices to read the symbol '$*$' in Eq. (2.1b) as an ordinary multiplication sign.

- Stochastic impacts between the Brownian particle and the constituents of the heat bath occur virtually uncorrelated.

- On a macroscopic level, the interaction between Brownian particle and heat bath is sufficiently well described by the constant viscous friction coefficient α and the stochastic Langevin force $L(t) = (2D)^{1/2} * \zeta(t)$.

- Eqs. (2.1) hold in the lab frame Σ, corresponding to the specific inertial system, where the average velocity of the heat bath particles vanishes for all times t.

In Section 2.2 we shall review how stochastic dynamical equations similar to Eqs. (2.1) can be derived and motivated by means of specific microscopic models.

In the mathematical literature [64, 66], SDEs like the Langevin Eq. (2.1b) are usually written in the differential notation

$$\mathrm{d}X(t) = (P/M)\,\mathrm{d}t, \tag{2.3a}$$
$$\mathrm{d}P(t) = -\alpha P\,\mathrm{d}t + (2D)^{1/2} * \mathrm{d}B(t). \tag{2.3b}$$

Here, $\mathrm{d}X(t) := X(t+\mathrm{d}t) - X(t)$ denotes the position increment, $\mathrm{d}P(t) := P(t+\mathrm{d}t) - P(t)$ the momentum increment; $B(t)$ is a standardized 1D Brownian motion or, equivalently, a standard Wiener process [41, 64, 66, 67], whose increments

$$\mathrm{d}B(t) := B(t+\mathrm{d}t) - B(t) \tag{2.3c}$$

are defined to be stochastically independent[3] and characterized a the Gaussian probability density function (PDF)

$$\mathbb{P}\{\mathrm{d}B(t) \in [y, y+\mathrm{d}y]\} = (2\pi\,\mathrm{d}t)^{-1/2} \exp\bigl[-y^2/(2\,\mathrm{d}t)\bigr]\,\mathrm{d}y; \tag{2.3d}$$

i.e., the increments $\mathrm{d}B(t)$ are independent random numbers drawn from a normal distribution with variance $\mathrm{d}t$. The two different representations (2.1) and (2.3) of the Ornstein-Uhlenbeck process may be connected by formally identifying

$$\mathrm{d}B(t) = \zeta(t)\mathrm{d}t. \tag{2.4}$$

In the remainder, SDEs will primarily be written in the differential notation of Eq. (2.3), which may also be viewed as a simple numerical integration scheme, cf., e.g., Ref. [79, 336]

[3] This means that the joint probability density of an arbitrary collection of subsequent increments $\mathrm{d}B(t_i)$ is a product of the Gaussians $\mathbb{P}[\mathrm{d}B(t_i)]$; see, e.g., [64, 66] for a precise mathematical definition.

and App. C. From Eq. (2.3d) and the independence of the increments at different times $s \neq t$, it follows that

$$\langle \mathrm{d}B(t) \rangle = 0, \qquad \langle \mathrm{d}B(t)\, \mathrm{d}B(s) \rangle = \begin{cases} 0, & t \neq s \\ \mathrm{d}t, & t = s, \end{cases} \tag{2.5}$$

where now the expectation $\langle \cdot \rangle$ is taken with respect to the probability measure of the Wiener process $B(t)$.

In order for Eqs. (2.3) to define a well-posed problem, they must be complemented by initial conditions.[4] Generally, one could consider either deterministic initial conditions by fixing $X(0) = x_0$ and $P(0) = p_0$ or probabilistic initial conditions by specifying initial distributions for $X(0)$ and $P(0)$. Then, the solutions of Eqs. (2.3) read explicitly

$$X(t) = X(0) + \int_0^t \mathrm{d}s\; P(s)/M, \tag{2.6a}$$

$$P(t) = P(0)\, e^{-\alpha t} + (2D)^{1/2} e^{-\alpha t} \int_0^t e^{\alpha s} * \mathrm{d}B(s). \tag{2.6b}$$

In the remainder, we primarily refer to deterministic initial conditions, assuming that the initial position $X(0) = x_0$ and the initial momentum $P(0) = p_0$ of the Brownian particle are known exactly. Combining the solution (2.6) with Eq. (2.3d), one finds for the first two moments of the momentum coordinate [36, 38]

$$\begin{aligned} \langle P(t) \rangle &= P(0)\, e^{-\alpha t}, \\ \left\langle P(t)^2 \right\rangle &= P(0)^2\, e^{-2\alpha t} + \frac{D}{\alpha}\, (1 - e^{-2\alpha t}), \end{aligned} \tag{2.7}$$

while the first centered moments of the position coordinate are obtained as

$$\langle X(t) - X(0) \rangle = \frac{P(0)}{\alpha M}\, (1 - e^{-\alpha t}), \tag{2.8a}$$

$$\begin{aligned} \left\langle [X(t) - X(0)]^2 \right\rangle = {} & \frac{2D\, t}{(\alpha M)^2} + \left[\frac{P(0)}{\alpha M} \right]^2 \left(1 - e^{-\alpha t} \right)^2 + \\ & \frac{D}{\alpha^3 M^2} \left(-3 + 4 e^{-\alpha t} - e^{-2\alpha t} \right). \end{aligned} \tag{2.8b}$$

The asymptotic spatial diffusion constant $\mathcal{D}_\infty$, not to be confused with the noise amplitude D, is usually defined by

$$2\mathcal{D}_\infty := \lim_{t \to \infty} \frac{1}{t}\, \left\langle [X(t) - X(0)]^2 \right\rangle. \tag{2.9}$$

[4] Without loss of generality we fix the initial time $t_0 = 0$.

From Eq. (2.8b) we find for the Ornstein-Uhlenbeck process the classical result

$$\mathcal{D}_\infty := D/(\alpha M)^2. \tag{2.10}$$

When studying SDEs of the type (2.3b), one is typically interested in the probability

$$f(t,x,p)\,\mathrm{d}x\,\mathrm{d}p$$

of finding the Brownian particle at time t in the infinitesimal phase space interval $[x, x+\mathrm{d}x]\times[p, p+\mathrm{d}p]$. The non-negative phase space PDF $f(t,x,p)\geq 0$ of the Brownian particle is normalized at all times, i.e.

$$1 = \int \mathrm{d}x\mathrm{d}p\, f(t,x,p)\,, \qquad\qquad \forall\, t>0; \tag{2.11}$$

where, here and below, unspecified integrals range over the full phase space, position space, or momentum space, respectively. Given the phase space PDF $f(t,x,p)$, the marginal momentum PDF $\phi(t,p)$ and the marginal position PDF $\varrho(t,x)$ are defined by

$$\phi(t,p) = \int \mathrm{d}x\, f(t,x,p), \tag{2.12a}$$

$$\varrho(t,x) = \int \mathrm{d}p\, f(t,x,p). \tag{2.12b}$$

Deterministic initial data $X(0)=x_0$ and $P(0)=p_0$ translate into the initial conditions

$$f(0,x,p) = \delta(x-x_0)\delta(p-p_0), \tag{2.13a}$$

$$\phi(0,p) = \delta(p-p_0), \tag{2.13b}$$

$$\varrho(0,x) = \delta(x-x_0). \tag{2.13c}$$

For the Ornstein-Uhlenbeck process from Eq. (2.3b), the Fokker-Planck equation (FPE) governing the momentum PDF $\phi(t,p)$ reads [220]

$$\frac{\partial\phi}{\partial t} = \frac{\partial}{\partial p}\left(\alpha p\phi + D\frac{\partial\phi}{\partial p}\right). \tag{2.14}$$

Adopting the deterministic initial condition (2.13b), the time-dependent solution of Eq. (2.14) is given by [36, 220]

$$\phi(t,p) = \left\{\frac{\alpha}{2\pi D[1-\exp(-2\alpha t)]}\right\}^{1/2} \exp\left\{-\frac{\alpha[p-p_0\exp(-\alpha t)]^2}{2D[1-\exp(-2\alpha t)]}\right\}. \tag{2.15}$$

In the limit $t\to\infty$ this solution reduces to the stationary Gaussian distribution

$$\phi_\infty(p) = \left(\frac{\alpha}{2\pi D}\right)^{1/2}\exp\left(-\frac{\alpha p^2}{2D}\right). \tag{2.16}$$

For a given momentum distribution $\phi(t,p)$ of the Brownian particle, the corresponding velocity PDF $\psi(t,v)$ is defined by

$$\psi(t,v) := \left|\frac{\mathrm{d}p}{\mathrm{d}v}\right| \; \phi(t,p(v)), \tag{2.17}$$

where $p = Mv$ in the nonrelativistic case. Hence, by imposing the *Einstein relation*

$$D = \alpha M k_{\mathrm{B}} \mathcal{T}, \tag{2.18}$$

the stationary momentum PDF (2.16) is seen to be equivalent to Maxwell's velocity distribution

$$\psi_{\mathrm{M}}(v) = \left(\frac{M}{2\pi k_{\mathrm{B}}\mathcal{T}}\right)^{1/2} \exp\left(-\frac{Mv^2}{2k_{\mathrm{B}}\mathcal{T}}\right), \tag{2.19}$$

where $\mathcal{T}$ is the temperature of the heat bath and k_{B} the Boltzmann constant. Moreover, the asymptotic spatial diffusion constant from Eq. (2.10) takes the form

$$\mathcal{D}_\infty := k_{\mathrm{B}}\mathcal{T}/(\alpha M). \tag{2.20}$$

The Einstein relation (2.18) represents the simplest example of a fluctuation-dissipation relation (FDR) by linking the noise amplitude D and the friction coefficient α to the temperature $\mathcal{T}$ of the heat bath. On the level of the Langevin description, this relation is motivated by the plausible assumption that, after a certain relaxation time, the Brownian particle will be in thermodynamic equilibrium with the surrounding bath. In Section 2.2 it will be discussed how generalized FDRs may arise from specific microscopic models for the interaction between Brownian particle and heat bath. Before doing this, however, we briefly address a few generalizations of the free Ornstein-Uhlenbeck process (2.3b).

Ornstein-Uhlenbeck process in an external force field A widely studied generalization of the free Ornstein-Uhlenbeck process (2.3b) corresponds to the case where an additional external force field acts on the Brownian particle [68]. Focussing as before on the 1D case, the generalized SDE for the Ornstein-Uhlenbeck process in an external force field $\mathcal{F}(t,x)$ reads[5]

$$\mathrm{d}X = (P/M)\,\mathrm{d}t, \tag{2.21a}$$

$$\mathrm{d}P = \mathcal{F}(t,X)\,\mathrm{d}t - \alpha\, P\,\mathrm{d}t + (2D)^{1/2} * \mathrm{d}B(t). \tag{2.21b}$$

Examples include external gravitational or electric forces. The FPE describing the phase space density $f(t,x,p)$ of the stochastic process (2.21) is given by

$$\frac{\partial f}{\partial t} + \frac{p}{M}\frac{\partial f}{\partial x} + \mathcal{F}(t,x)\frac{\partial f}{\partial p} = \frac{\partial}{\partial p}\left(\alpha p f + D\frac{\partial f}{\partial p}\right). \tag{2.22}$$

[5]More generally, one could also consider momentum (i.e., velocity) dependent force fields as, e.g., the Lorentz force in three space dimensions.

For arbitrary time and position dependent force fields $\mathcal{F}(t,x)$ it is generally very difficult, and in many cases even impossible, to find exact time-dependent solutions of the Fokker-Planck equation (2.22). In the simpler case of a time-independent, conservative force field $\mathcal{F}(t,x) \equiv F(x)$ with confining[6] potential $\Phi(x)$, i.e.

$$F(x) = -\frac{\partial}{\partial x}\Phi(x), \tag{2.23}$$

one can determine the stationary solution attained in the limit $t \to \infty$. Imposing as above the Einstein relation $D = \alpha M k_{\mathrm{B}} \mathcal{T}$, the stationary solution of Eq. (2.22) is given by the Maxwell-Boltzmann distribution [220, 221]

$$f(x,p) = \mathcal{Z}^{-1} \exp\left\{ -\beta \left[\frac{p^2}{2M} + \Phi(x) \right] \right\}, \qquad \beta := (k_{\mathrm{B}}\mathcal{T})^{-1}, \tag{2.24}$$

where the normalization constant $\mathcal{Z}$ is determined by Eq. (2.11).

Another important class of applications includes time periodic force fields, satisfying $\mathcal{F}(t,x) = \mathcal{F}(t+\Delta t, x)$ for some fixed period Δt. In this case it is sometimes possible to derive approximate asymptotic solutions of the FPE (2.22) by considering the limit $t \to \infty$. These asymptotic solutions are usually also time periodic and can exhibit phase shifts. They may give rise to a number of interesting phenomena such as, e.g., stochastic resonance [408–416].

2.1.2 Nonlinear Langevin equations

In the case of the classical Ornstein-Uhlenbeck process (2.3) the interaction between Brownian particle and heat bath is modeled by means of a constant friction coefficient α and a constant noise amplitude D. Although this approximation has proven to be useful for many problems, it becomes inappropriate in several other cases; e.g., if the friction force increases nonlinearly with the velocity of the Brownian particle. Accordingly, a simple generalization of the free Ornstein-Uhlenbeck process (2.3) is obtained by considering momentum dependent coefficient functions $\alpha(p)$ and/or $D(p)$, leading to the so-called *nonlinear* Langevin equation [24, 101, 417]

$$\mathrm{d}P = -\alpha_{\bullet}(P)\, P\, \mathrm{d}t + [2D(P)]^{1/2} \bullet \mathrm{d}B(t). \tag{2.25}$$

Here, the symbol '$\bullet$' signals the post-point discretization interpretation [60] of the SDE (2.25), which means that the coefficient function $D(p)$ is evaluated at the post-point $P(t+\mathrm{d}t)$. A stochastic force with nonlinearly momentum dependent noise amplitude function $D(p)$ as in Eq. (2.25) is usually referred to as 'multiplicative' noise, in contrast

[6] Conventionally, a potential $\Phi(x)$ is called 'confining' if it increases sufficiently fast for $|x| \to \infty$ so that the phase space PDF f is normalizable.

to the 'additive' noise encountered in Eqs. (2.3) and (2.21). When considering SDEs that contain multiplicative noise terms, the specification of the discretization rule is necessary because of the fact that, for fixed functions $\alpha(p)$ and $D(p)$, different discretization schemes in general lead to nonequivalent stochastic processes; put differently, the values of the stochastic integral $P(t)$ defined by Eq. (2.25) depend on the choice of discretization rule. This is the most essential difference compared with ordinary differential equations, whose integral curves (i.e., solutions) are independent of the underlying discretization scheme when taking the continuum limit $\mathrm{d}t \to 0$.

In Eq. (2.25) we opted for the post-point rule; in principle, other discretization rules can be used as well [65, 418, 419]. The most prominent alternatives are Ito's [50, 51] pre-point discretization ($*$), corresponding to computing function $D(p)$ at $P(t)$, and the mid-point rule ($\circ$) of Stratonovich [57–59] and Fisk [55, 56], where D is evaluated at the mean value $[P(t) + P(t + \mathrm{d}t)]/2$. From the mathematical point of view, the choice of the discretization rule reduces to a matter of convenience due to the following fact: For each pair of sufficiently smooth functions $(\alpha_\bullet(p), D(p))$, one can determine a pair of functions $(\alpha_{\circ|*}(p), D(p))$ which describes exactly the same stochastic dynamics when combined with another discretization rule $\circ$ and $*$, respectively. The corresponding conversion formulae are summarized in App. C.

From the practical point of view, each of the three above mentioned discretization methods possesses its own merits and drawbacks: Ito's pre-point rule ($*$) is particularly convenient for numerical simulations, but care is required when considering nonlinear transformations $G(P)$ of the momentum coordinate due to modifications of the differential calculus, cf. App. C. By contrast, if one adopts the Stratonovich-Fisk mid-point rule ($\circ$), then the transformation rules from ordinary differential calculus carry over, but it becomes more difficult to implement this mid-point rule in numerical simulations. The latter disadvantage also applies to the post-point rule employed in Eq. (2.25). However, as we shall see next, the post-point rule ($\bullet$) leads to a particularly simple form of the FDR.

Adopting the post-point rule, the Fokker-Planck equation for the momentum PDF $\phi(t, p)$ of the stochastic process defined by Eq. (2.25) reads

$$\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial p}\left[\alpha_\bullet(p) p\, \phi + D(p) \frac{\partial \phi}{\partial p}\right]. \tag{2.26}$$

Its stationary solution is given by[7]

$$\phi_\infty(p) = \mathcal{N} \exp\left[-\int_{-p_*}^{p} \mathrm{d}p'\, \frac{\alpha_\bullet(p')}{D(p')}\, p'\right], \tag{2.27}$$

[7] If we had considered Eq. (2.25) with another stochastic integral interpretation (e.g., pre-point or mid-point discretization), then the corresponding FPE would be different from Eq. (2.26), cf. App. C; accordingly, one would also obtain another stationary distribution.

where $\mathcal{N}$ is a normalization constant, and p_* some arbitrary constant such that the integral in the exponential exists.

As follows from the general form (2.27) of the stationary solution, one may generate arbitrary momentum distributions (e.g., Maxwell, Bose, Fermi or power law distributions) by choosing the friction and noise amplitude functions $\alpha_\bullet(p)$ and $D(p)$ in a suitable manner [12, 420]. To briefly illustrate this, consider some normalized target PDF $\phi_*(p) \geq 0$. We would like to fix the relation between $\alpha_\bullet$ and D such that the stationary solution $\phi_\infty(p)$ coincides with $\phi_*(p)$. Equating $\phi_*(p)$ with $\phi_\infty(p)$ from Eq. (2.27), taking the logarithm and differentiating with respect to p we find the condition

$$\frac{\alpha_\bullet(p)}{D(p)}\, p = -\frac{\mathrm{d}}{\mathrm{d}p} \log \phi_*(p). \tag{2.28}$$

In particular, by imposing the generalized Einstein relation [24, 101, 102]

$$\frac{\alpha_\bullet(p)}{D(p)} = (M k_\mathrm{B} \mathcal{T})^{-1}, \tag{2.29}$$

the stationary distribution $\phi_\infty(p)$ reduces to the Maxwell distribution from Eq. (2.16). It should be stressed, however, that the FDRs (2.28) and (2.29) do fix only one of the two coefficients $\alpha_\bullet(p)$ and $D(p)$. Put differently, one is still free to adapt, e.g., the function $\alpha_\bullet(p)$ such that the stochastic process (2.25) exhibits the correct relaxation behavior. This freedom is a main reason why the Langevin approach is successfully applicable to a wide range of thermalization processes [101]. Physically reasonable expressions for $\alpha_\bullet(p)$ may be obtained from kinetic theory [308, 310, 421–424] or microscopic Hamiltonian models that take into account the interactions as well as the statistical properties of the heat bath [13, 80–83, 86, 425]. Examples will be discussed in Section 2.2.

Langevin equations of the type (2.25) and nonlinear friction effects [426] have been studied extensively in various contexts during the past decades (see, e.g., the review by Klimontovich [101]). The applications cover a wide range of different areas including laser physics [101, 102, 427], optical lattices [428, 429], plasma physics [430–433], high energy physics [308, 310], biologically and chemically motivated population and reaction dynamics [434], active Brownian motion models [424, 435–440], or theoretical and experimental studies of excitation and transition phenomena in nonlinear systems [441–444].

However, with regard to our subsequent discussion of relativistic Brownian motions, it will be most important to keep in mind that the nonlinear Langevin equation (2.25) provides a tool for constructing Brownian motion processes with arbitrary stationary velocity and momentum distributions [12, 420].

2.1.3 Other generalizations

Thus far we have focussed on two of the most commonly considered examples of nonrelativistic Brownian motion processes, the classical linear Ornstein-Uhlenbeck process (2.1) and its nonlinear counterpart (2.25). Their generalization within the framework of special relativity shall be our main concern in Chapter 4. At this point, however, it may also be useful to briefly address further possible modifications of Eqs. (2.1) and (2.25) that have been studied in the context of nonrelativistic physics during the past years, and whose generalizations to special relativity present open problems for the future.

The stochastic processes defined by Eqs. (2.1) and (2.25) share the property that the underlying noise source is modeled by a standard Wiener process $B(t)$. Generally, it is also possible to consider other driving processes such as Poisson processes [64, 66] or Lévy processes [445–447], which may give rise to so-called anomalous super- or sub-diffusion effects; see, e.g., the reviews by Bouchaud and Georges [105] and Metzler and Klafter [106].[8] Moreover, one may abandon the assumption (2.2b) of δ-correlated 'white' noise by considering stochastic processes that are driven by colored noise. For example, one can replace Eq. (2.2b) with

$$\langle \zeta(t)\, \zeta(s) \rangle = \frac{1}{\tau_{\mathrm{n}}} \exp(-|t-s|/\tau_{\mathrm{n}}), \tag{2.30}$$

where the parameter τ_{n} is the relaxation time of the driving noise ζ. The mathematical analysis of processes driven by colored noise is considerably more complicated than in the case of δ-correlated white noise; for a detailed discussion we refer to the review by Hänggi and Jung [103].

2.2 Microscopic models

When considering Langevin equations of the type (2.1) and (2.25), one may in principle distinguish between the two following tasks:

a) One can postulate the Langevin equation as a phenomenological model equation, study the mathematical consequences and compare these predictions with experiments in order to (in)validate the theory. Adopting this approach, the parameters and the explicit functional form of the friction and noise amplitude functions have to be determined from experimental data [451].

[8] Anomalous diffusion processes [448–450] are characterized by an asymptotic spatial mean square displacement that grows proportional to t^{γ}, with $\gamma > 1$ and $\gamma < 1$ corresponding super- and sub-diffusion, respectively [106, 445].

b) Alternatively, one can try to motivate and derive Langevin equations from microscopic models. If successful, this approach yields explicit expressions for the friction and noise functions in terms of the microscopic model parameters.

The remainder of this section addresses the latter problem, which has attracted considerable interested over the past decades [13,80–86,294,326,423,425,452]. From the physicist's point of view, Langevin equations provide an approximate description of the 'exact' microscopic dynamics. Hence, in order to derive SDEs like (2.1) or (2.25) from, e.g., microscopic Hamiltonian mechanics one has to impose certain approximations. These approximations determine the range of applicability of the Langevin approach. Generally, one can pursue at least two different routes for deriving SDEs of the type (2.1) and (2.25) from more precise models:

(i) Starting from a Boltzmann equation [267, 268, 290] or master equation [65] for the one-particle probability density of the Brownian particle, one can try to reduce these integro-differential equations to a Fokker-Planck equation by performing suitable approximations [294,308,310,326,422–424,452]. Once the Fokker-Planck equation has been found, it is straightforward to write down a corresponding Langevin dynamics [65, 66]. The microscopic collision dynamics is then encoded in the scattering cross-sections appearing in the collision integral of the Boltzmann equation [268, 290].

(ii) Alternatively, one may start from a microscopic (e.g., Hamiltonian) model describing the interaction between Brownian particle and heat bath. After eliminating the heat bath degrees of freedom from the equations of motion for the Brownian particle [13,80–86,425], one obtains a generalized Langevin equation which may be reduced to the form (2.1) and (2.25) in certain limit cases. As a byproduct, the FDRs arise naturally within this approach upon assuming a probability distribution for the (initial) bath configuration. To illustrate this by example, we next consider the oscillator model [80–86] and the elastic binary collision model [13]. In contrast to the more frequently studied harmonic oscillator model, the collision model from Section 2.2.2 can be extended to the relativistic case; cf. discussion in Section 4.3.

2.2.1 Harmonic oscillator model

The harmonic oscillator model represents the classic paradigm for constructing a generalized Langevin equation from a Hamiltonian model [68, 80–86]. The Hamiltonian function upon which the derivation is based reads [86]

$$H = \frac{P^2}{2M} + \Phi(X) + \sum_r \left[\frac{p_r^2}{2m_r} + \frac{m_r\omega_r^2}{2}\left(x_r - \frac{c_r}{m_r\omega_r^2} X \right)^2 \right]. \tag{2.31}$$

Here M, X and P are the mass, position and momentum of the Brownian particle and $\Phi(x)$ is an external potential field; x_r and p_r denote the position and momentum of a heat bath

particle with mass m_r, oscillator frequency ω_r and coupling constant c_r. Equation (2.31) yields the following Hamilton equations of motions:

$$M\dot{X} = P, \qquad \dot{P} = F(X) + \sum_r c_r \left(x_r - \frac{c_r}{m_r\omega_r^2} X \right); \tag{2.32a}$$

$$m_r\dot{x}_r = p_r, \qquad \dot{p}_r = -m_r\omega_r^2 x_r + c_r X\ , \qquad r = 1, \ldots, N, \tag{2.32b}$$

where $F(x) = -\mathrm{d}\Phi(x)/\mathrm{d}x$ is the conservative external force acting on the Brownian particle. As evident from Eqs. (2.32), Brownian particle and heat bath are coupled via linear forces in this model. By formally integrating Eqs. (2.32b) and inserting the solutions into Eq. (2.32a), one may eliminate the heat bath coordinates from Eqs. (2.32a), yielding the exact *generalized* Langevin equations [83, 86]

$$M\dot{X} = P, \tag{2.33a}$$

$$\dot{P} = F(X) - \int_0^t \mathrm{d}s\ \nu(t-s)P(s) + L(t), \tag{2.33b}$$

where, for given initial values $X(0), P(0), \{x_r(0), p_r(0)\}$, the memory friction kernel $\nu(t-s)$ and the Langevin noise force $L(t)$ are given by [86]

$$\nu(t-s) := \frac{1}{M}\sum_r \frac{c_r^2}{m_r\omega_r^2}\cos[\omega_r(t-s)], \tag{2.33c}$$

$$L(t) := \sum_r c_r \left\{ \left[x_r(0) - \frac{c_r}{m_r\omega_r^2} X(0) \right] \cos(\omega_r t) + \frac{p_r(0)}{m_r\omega_r}\sin(\omega_r t) \right\}. \tag{2.33d}$$

In order to be able to characterize the properties of the noise force $L(t)$ by means of an FDR, one still needs to impose a distribution for the initial conditions $\{x_r(0), p_r(0)\}$ of the bath variables. In principle, this initial distribution can be chosen arbitrarily. Of particular interest in canonical thermostatistics are equilibrium distributions of the Maxwell-Boltzmann type. In the case of the generalized Langevin equation (2.33), a plausible choice for the initial bath distribution corresponds to the PDF

$$f_{\mathrm{b}}(\{x_r(0), p_r(0)\}\,|\,X(0)\,) = \mathcal{Z}^{-1} \times \exp\left\{ -\beta \sum_r \left[\frac{p_r(0)^2}{2m_r} + \frac{m_r\omega_r^2}{2}\left(x_r(0) - \frac{c_r}{m_r\omega_r^2}X(0) \right)^2 \right] \right\}; \tag{2.34}$$

$\beta = (k_{\mathrm{B}}\mathcal{T})^{-1}$ is the inverse thermal energy, $\mathcal{T}$ the temperature, and $\mathcal{Z}$ the normalization constant. The initial position $X(0)$ of the Brownian particle enters in Eq. (2.34) as an independent parameter, i.e., averages with respect to $f_{\mathrm{b}}(\{x_r(0), p_r(0)\}\,|\,X(0))$ are conditional on the initial Brownian particle position $X(0)$. Averaging the stochastic force $L(t)$ with respect to f_{b} from Eq. (2.34), one finds

$$\langle L(t)\rangle_{\mathrm{b}} = 0, \tag{2.35a}$$

$$\langle L(t)L(s)\rangle_{\mathrm{b}} = Mk_{\mathrm{B}}\mathcal{T}\,\nu(t-s). \tag{2.35b}$$

Equation (2.35b) represents the FDR for the generalized Langevin equation (2.33) given the initial bath distribution (2.34). The generalized Langevin equation (2.33) differs from Eqs. (2.1), (2.21) and (2.25) through the memory friction $\nu(t-s)$. The SDE (2.21), which describes the Ornstein-Uhlenbeck process in an external force field, is recovered from Eqs. (2.33) in the limit case[9]

$$\nu(t-s) = 2\alpha\,\delta(t-s), \tag{2.36}$$

where α is a constant friction coefficient. The limit case (2.36) can be illustrated by rewriting the friction kernel (2.33c) in the more general form

$$\nu(t-s) = \int_0^\infty \mathrm{d}\omega\; C(\omega)\;\cos[\omega(t-s)]. \tag{2.37}$$

By fixing the amplitude function $C(\omega)$ as

$$C(\omega) = \frac{1}{M}\sum_r \frac{c_r^2}{m_r\omega_r^2}\;\delta(\omega-\omega_r), \tag{2.38}$$

one recovers the memory friction (2.33c) as a special case of Eq. (2.37). In order to obtain the limit case (2.36) from Eq. (2.37), one can use the cosine-decomposition of the Dirac δ-function

$$\begin{aligned}\delta(t-s) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\omega\; e^{i\omega(t-s)}\\ &= \frac{1}{\pi}\int_0^\infty \mathrm{d}\omega\;\cos[\omega(t-s)].\end{aligned} \tag{2.39}$$

Hence, upon comparing Eqs. (2.39) and (2.37), the white noise limit (2.36) corresponds to the choice

$$C(\omega) = (2\alpha)/\pi. \tag{2.40}$$

The harmonic oscillator model[10] provides a useful microscopic justification for the Langevin equations (2.1) and (2.21) of the Ornstein-Uhlenbeck process. Unfortunately, this model cannot be transferred to special relativity, as it is based on instantaneous harmonic interactions-at-distance which violate fundamental relativistic principles. Therefore, in the last part of this chapter we shall consider another microscopic model which is based on strictly localized elastic binary collisions and, thus, can be extended to special relativity.

[9]The prefactor 2 is required in Eq. (2.36) because of the convention $\int_0^t \mathrm{d}s\;\delta(t-s)P(s) = P(t)/2$.

[10]The quantum mechanical generalization of the harmonic oscillator model represents a paradigm for quantum Brownian motions and has been studied, e.g., in [82, 87–93, 99]; see also the reviews by Grabert et al. [107] and Hänggi and Ingold [98].

2.2.2 Elastic binary collision model

The 1D elastic binary collision model [13] is based on the idea that the stochastic motion of a Brownian particle (mass M) is caused by frequent elastic collisions with smaller heat bath particles (mass $m \ll M$).[11] As before, we denote the coordinates and momenta of the heat bath particles by $\{x_r, p_r\}$, where $r = 1, \ldots, N$ and $N \gg 1$.

Collision kinematics We consider the elastic collision of the Brownian particle (momentum P, kinetic energy E) with a heat bath particle (momentum p_r, kinetic energy ϵ_r). The collision process is governed by the energy-momentum conservation laws

$$E + \epsilon_r = \hat{E} + \hat{\epsilon}_r, \tag{2.41a}$$

$$P + p_r = \hat{P} + \hat{p}_r, \tag{2.41b}$$

where hat-symbols refer to the state after the collision. In the nonrelativistic case, we have, e.g., before the collision

$$P = MV, \qquad E = P^2/(2M), \tag{2.42a}$$

$$p_r = mv_r, \qquad \epsilon_r = p_r^2/(2m) \tag{2.42b}$$

with V and v_r denoting the velocities. Taking into account the kinematic conservation laws (2.41), we find that the momentum gain ΔP_r of the Brownian particle per single collision is given by

$$\Delta P_r := \hat{P} - P = -\frac{2m}{M+m}\, P + \frac{2M}{M+m}\, p_r. \tag{2.43}$$

In order to construct a Langevin-like equation from Eqs. (2.41)–(2.43), we consider the total momentum change $\delta P(t)$ of the Brownian particle within the time interval $[t, t+\delta t]$, assuming that:

- collisions occurring within $[t, t+\delta t]$ can be viewed as independent events;
- the time step δt is sufficiently small, so that

  $$|\delta P(t)/P(t)| \ll 1$$

  holds true. In particular, δt is supposed to be so small that there occurs at most only one collision between the Brownian particle and a specific heat bath particle r; on the other hand, δt should still be large enough, so that the total number of collisions within δt is larger than 1. These requirements can be fulfilled simultaneously only if $m \ll M$ holds.

[11] Similar approaches are known from unimolecular rate theory, see, e.g., Section V in [68]. In the context of quantum Brownian motions, a quantum-mechanical version of the collision model was proposed and studied by Pechukas [95], and Tsonchev and Pechukas [96].

With these two assumptions, we can approximate

$$\delta P(t) := P(t+\delta t) - P(t) \approx \sum_{r=1}^{N} \Delta P_r \; I_r(t,\delta t), \tag{2.44}$$

where $I_r(t,\delta t) \in \{0,1\}$ is the indicator function for a collision with the heat bath particle r during the interval $[t, t+\delta t]$; i.e.

$$I_r(t,\delta t) = \begin{cases} 1 & \text{if a collision has occurred,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.45}$$

Evidently, the collision indicators depend on the position and velocity coordinates of the collision partners. In the 1D case, $I_r(t,\delta t)$ can be expressed in the form[12]

$$\begin{aligned} I_r(t,\delta t) \;\; = \;\; & \Theta(X - x_r)\,\Theta(x_r' - X')\,\Theta(v_r - V) + \\ & \Theta(x_r - X)\,\Theta(X' - x_r')\,\Theta(V - v_r), \end{aligned} \tag{2.46}$$

where $X = X(t)$, $x_r = x_r(t)$ are the 'initial' positions of the colliding particles at time t, and

$$X' = X + V\,\delta t, \qquad\qquad x_r' = x_r + v_r\,\delta t$$

their projected positions at time $t+\delta t$. The collision indicator from Eq. (2.46) is characterized by

$$I_r(t,0) \;\; = \;\; 0, \tag{2.47a}$$

$$[I_r(t,\delta t)]^j \;\; = \;\; I_r(t,\delta t), \qquad\qquad j = 1,2,\ldots; \tag{2.47b}$$

and the Taylor-expansion of I_r at $\delta t = 0$ reads [13]

$$I_r(t,\delta t) \;\; \approx \;\; \frac{\delta t}{2}\,|v_r - V|\,\delta(x_r - X). \tag{2.48a}$$

Combining Eqs. (2.43), (2.44) and (2.48a) yields

$$\delta P(t) \;\; \approx \;\; -2\left[\sum_{r=1}^{N} \frac{m}{M+m}\, I_r(t,\delta t)\right] P(t) + 2\sum_{r=1}^{N} \frac{M}{M+m}\, p_r\, I_r(t,\delta t), \tag{2.48b}$$

[12] The Heaviside-function $\Theta(x)$ is defined as the integral over the Dirac δ-function, i.e., $\Theta(x) := 0, x < 0$; $\Theta(0) := 1/2$; $\Theta(x) := 1, x > 0$. When considering higher space dimensions, the expression (2.46) for the indicator function has to be modified accordingly, e.g., by taking into account the geometric shape of the Brownian particle.

where, additionally, it was assumed that for each collision occurring within $[t, t+\delta t]$, the momentum of the Brownian particle before the collision is approximately equal to the 'initial' value $P(t)$. In view of $m \ll M$, Eq. (2.48b) can be simplified further to give

$$\delta P(t) \;\approx\; -2\left[\sum_{r=1}^{N} \frac{m}{M}\, I_r(t,\delta t)\right] P(t) + 2\sum_{r=1}^{N} p_r\, I_r(t,\delta t). \tag{2.48c}$$

A comparison with the Langevin equations (2.3b) and (2.25) suggests that, heuristically, the first term on the rhs. of Eq. (2.48c) can be interpreted as a 'friction' term, while the second term represents 'noise'. On the other hand, although looking quite similar to a Langevin equation, Eq. (2.48c) is still considerably more complicated than, e.g., the nonlinear Langevin equation (2.25). This is due to the fact that the collision indicators $I_r(t,\delta t)$ from Eq. (2.48a) depend not only on the Brownian particle's position and velocity but also on the stochastic bath variables $\{x_r, v_r\}$. Nevertheless, it is possible to calculate the statistical properties of the momentum increments $\delta P(t)$ from Eqs. (2.48), provided one specifies a distribution for the heat bath particles.

Bath distribution and drift In principle, one can use Eqs. (2.48) to calculate the statistical moments $\langle(\delta P)^j\rangle_{\rm b}$ for an arbitrarily given heat bath PDF $f_{\rm b}^N(\{x_r,p_r\})$. Here, we shall focus on the situation where the (infinitely large) heat bath is given by a quasi-ideal gas which is in thermal equilibrium with its environment. In this case, the one-particle PDF $f_{\rm b}^1(x_r,p_r)$ is given by the spatially homogeneous Maxwell distribution

$$f_{\rm b}^1(x_r,p_r) \;=\; (2\pi m k_{\rm B}\mathcal{T})^{-1/2}\, L^{-1}\, \exp\!\left(-\frac{p_r^2}{2mk_{\rm B}\mathcal{T}}\right), \tag{2.49}$$

where $x_r \in [0,L]$ with L being the 1D container volume. Moreover, we will assume that:

- the heat bath particles are *independently* and *identically* distributed;
- the collisions with the Brownian particle do not significantly alter the bath distribution, so that the total energy of the bath particles remains constant.

The above assumptions can be justified for a sufficiently large bath, if collisions between the bath particles rapidly reestablish a spatially homogeneous bath distribution.

In order to calculate the moments $\langle(\delta P)^j\rangle_{\rm b}$, we note that, for a spatially uniform bath distribution as in Eq. (2.49), the one-particle expectation value $\langle G(x_r,v_r)\,[I_r(t,\delta t)]^j\rangle_{\rm b}$ is given by

$$\begin{aligned}\big\langle G(x_r,v_r)\,[I_r(t,\delta t)]^j\big\rangle_{\rm b} &\overset{(2.47a)}{=} \langle G(x_r,v_r)\, I_r(t,\delta t)\rangle_{\rm b} \\ &\overset{(2.48a)}{\approx} \frac{\delta t}{2L}\int_{-\infty}^{\infty}\!{\rm d}v_r\; G(X,v_r)\,|v_r-V|\,\psi_{\rm b}(v_r),\end{aligned} \tag{2.50}$$

with $\psi_{\rm b}(v_r)$ denoting the one-particle velocity PDF of the heat bath particles. For the canonical heat bath distribution from Eq. (2.49), the velocity PDF $\psi_{\rm b}(v)$ corresponds to the Maxwellian

$$\psi_{\rm b}(v_r) \;=\; \left(v_{\rm B}^2\pi\right)^{-1/2}\exp\!\left(-v_r^2/v_{\rm B}^2\right), \qquad\qquad v_{\rm B} := (2k_{\rm B}\mathcal{T}/m)^{1/2}. \tag{2.51}$$

In particular, we obtain

$$\begin{aligned}
N\,\langle I_r(t,\delta t)\rangle_{\rm b} &= \frac{n_{\rm b}v_{\rm B}}{2}\left\{\pi^{-1/2}\exp\!\left[-\left(\frac{P}{p_{\rm B}}\right)^2\right]+\left(\frac{P}{p_{\rm B}}\right)\mathrm{erf}\!\left(\frac{P}{p_{\rm B}}\right)\right\}\delta t,\\
N\,\langle p_r\, I_r(t,\delta t)\rangle_{\rm b} &= -\frac{n_{\rm b}v_{\rm B}}{4}\,p_{\rm B}\left(\frac{m}{M}\right)\mathrm{erf}\!\left(\frac{P}{p_{\rm B}}\right)\delta t,\\
N\,\left\langle p_r^2\, I_r(t,\delta t)\right\rangle_{\rm b} &= \frac{n_{\rm b}v_{\rm B}}{2}\,p_{\rm B}^2\left(\frac{m}{M}\right)^2\left\{\pi^{-1/2}\exp\!\left[-\left(\frac{P}{p_{\rm B}}\right)^2\right]+\frac{1}{2}\left(\frac{P}{p_{\rm B}}\right)\mathrm{erf}\!\left(\frac{P}{p_{\rm B}}\right)\right\}\delta t,
\end{aligned} \tag{2.52}$$

where $n_{\rm b} = N/L$ is the number density of the heat bath particles, $p_{\rm B} := Mv_{\rm B} = M(2k_{\rm B}\mathcal{T}/m)^{1/2}$, and the error function $\mathrm{erf}(z)$ is defined by

$$\mathrm{erf}(z) := \frac{2}{\sqrt{\pi}}\int_0^z \mathrm{d}x\; e^{-x^2}.$$

By making use of Eqs. (2.52), we find the *mean drift* of the collision model:

$$\begin{aligned}
\langle\delta P(t)\rangle_{\rm b} &= -2\left(\frac{m}{M}\right)\sum_{r=1}^{N}\langle I_r(t,\delta t)\rangle_{\rm b}\;P + 2\sum_{r=1}^{N}\langle p_r\; I_r(t,\delta t)\rangle_{\rm b}\\
&= -2N\left(\frac{m}{M}\right)\langle I_r(t,\delta t)\rangle_{\rm b}\;P + 2N\,\langle p_r\; I_r(t,\delta t)\rangle_{\rm b}\\
&\overset{(2.52)}{\approx} -2n_{\rm b}\,k_{\rm B}\mathcal{T}\left\{\pi^{-1/2}\exp\!\left[-\left(\frac{P}{p_{\rm B}}\right)^2\right]+\left(\frac{P}{p_{\rm B}}\right)\mathrm{erf}\!\left(\frac{P}{p_{\rm B}}\right)\right\}\left(\frac{P}{p_{\rm B}}\right)\delta t -\\
&\qquad n_{\rm b}\,k_{\rm B}\mathcal{T}\,\mathrm{erf}\!\left(\frac{P}{p_{\rm B}}\right)\delta t.
\end{aligned} \tag{2.53}$$

Figure 2.1 depicts the mean drift force $\langle\delta P(t)/\delta t\rangle_{\rm b}$, obtained from Eq. (2.53). The absolute value of this drift force grows linearly for small momentum values (Ornstein-Uhlenbeck regime) and quadratically for large momentum values.

We still consider the second moment:

$$\begin{aligned}
\left\langle[\delta P(t)]^2\right\rangle_{\rm b} &= 4\left(\frac{m}{M}\right)^2\sum_{s=1}^{N}\sum_{r=1}^{N}\langle I_rI_s\rangle_{\rm b}\;P^2 - 8\left(\frac{m}{M}\right)\sum_{s=1}^{N}\sum_{r=1}^{N}\langle p_r\,I_rI_s\rangle_{\rm b}\;P +\\
&\qquad 4\sum_{r=1}^{N}\sum_{s=1}^{N}\langle p_rp_s\;I_r\;I_s\rangle_{\rm b}\,.
\end{aligned}$$

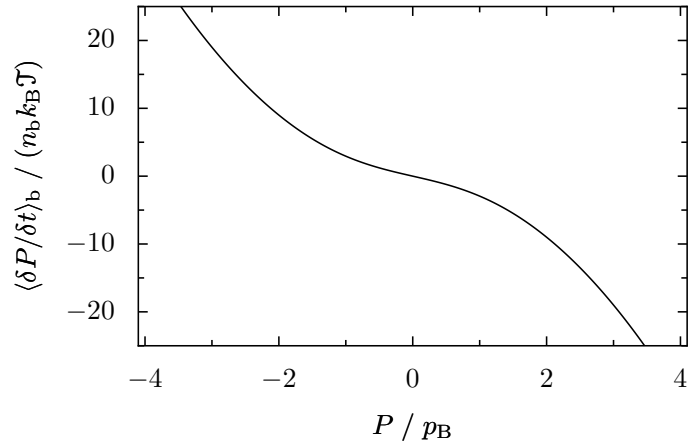


Figure 2.1: Nonrelativistic binary collision model. Mean drift force $\langle \delta P(t)/\delta t \rangle_{\rm b}$ from Eq. (2.53) with $n_{\rm b} = N/L$ denoting the number density of the heat bath particles and $p_{\rm B} := M(2k_{\rm B}\mathcal{T}/m)^{1/2}$ the characteristic momentum of a Brownian particle (mass M), surrounded by heat bath particles of mass m and temperature $\mathcal{T}$.

Neglecting contributions of order δt^2, only the terms with $s = r$ remain, and we obtain

$$\left\langle [\delta P(t)]^2 \right\rangle_{\rm b} \;\approx\; 4N \left(\frac{m}{M}\right)^2 \langle I_r \rangle_{\rm b} \; P^2 - 8N \left(\frac{m}{M}\right) \langle p_r \, I_r \rangle_{\rm b} \; P + 4N \left\langle p_r^2 \, I_r \right\rangle_{\rm b},$$

where the averages are determined by Eqs. (2.52). Evidently, higher moments $\left\langle [\delta P(t)]^j \right\rangle_{\rm b}$, $j \geq 3$ can be estimated in the same manner.

In the remainder of this section, however, we are going to discuss a systematic procedure for approximating the model equations (2.48) by a nonlinear SDE of the Langevin-type (2.25).

Langevin approximation As stated before, the Eq. (2.48c) for the momentum increments $\delta P(t)$ in the binary collision model is not yet a Langevin equation. Therefore, we conclude this section by discussing how one can approximate Eqs. (2.48) by a nonlinear SDE of the form [cf. Eq (2.25)]

$$\mathrm{d}P(t) = -\alpha(P)\, P \,\mathrm{d}t + [2D(P)]^{1/2} \bullet \mathrm{d}B(t). \tag{2.54}$$

Langevin equations of the type (2.54) are phenomenological model equations that provide a simplified description of the 'exact' microscopic dynamics. In order to obtain a useful Langevin model for a given microscopic dynamics, the coefficient functions α and D in Eq. (2.54) have to be chosen such that they yield the best possible approximation within this class of SDEs. Here, we define the 'best approximation' by means of the following two criteria: The stochastic process described by Eq. (2.54) should

- approach the correct stationary momentum distribution for the Brownian particle;
- be characterized by the same mean relaxation (drift) behavior as Eq. (2.48c).

The first criterion is equivalent to imposing the appropriate fluction-dissipation relation on the functions α and D. For the elastic collision model considered here, the expected stationary momentum PDF is given by the Maxwell distribution

$$\phi_\infty(p) = (2\pi M k_\mathrm{B}\mathcal{T})^{-1/2} \exp\bigl(-p^2/[2Mk_\mathrm{B}\mathcal{T}]\bigr). \tag{2.55}$$

According to the discussion in Section 2.1.2, this implies that α and D must be coupled by the Einstein condition

$$D(P) = \alpha(P)\, M k_\mathrm{B}\mathcal{T}. \tag{2.56}$$

The second (drift) criterion can be expressed mathematically as[13]

$$\left\langle \frac{\mathrm{d}P(t)}{\mathrm{d}t} \;\middle|\; P(t)=p \right\rangle \;\stackrel{!}{=}\; \left\langle \frac{\delta P(t)}{\delta t} \;\middle|\; P(t)=p \right\rangle_\mathrm{b}. \tag{2.57}$$

The rhs. of Eq. (2.57) may be determined from Eq. (2.53), yielding the mean drift force

$$\begin{aligned} g(p) \;&:=\; \left\langle \frac{\delta P(t)}{\delta t} \;\middle|\; P(t)=p \right\rangle_\mathrm{b} \\ &=\; -2\, n_\mathrm{b} k_\mathrm{B}\mathcal{T} \left\{ \pi^{-1/2} \left(\frac{p}{p_\mathrm{B}}\right) \exp\left[-\left(\frac{p}{p_\mathrm{B}}\right)^2\right] + \right. \\ &\qquad\qquad \left. \left[\left(\frac{p}{p_\mathrm{B}}\right)^2 + \frac{1}{2}\right] \operatorname{erf}\left(\frac{p}{p_\mathrm{B}}\right) \right\}. \end{aligned} \tag{2.58a}$$

In order to evaluate the lhs. of Eq. (2.57), we note that for the post-point discretization rule (•) it is known that [cf. (C.25)]

$$\bigl\langle [2D(P)]^{1/2} \bullet \mathrm{d}B(t) \mid P(t)=p \bigr\rangle \;=\; D'(p)\, \mathrm{d}t,$$

where the prime denotes the derivative with respect to the momentum variable. Substituting the Einstein relation (2.56), i.e., $D(p) = \alpha(P) M k_\mathrm{B}\mathcal{T}$, we obtain

$$\bigl\langle [2D(P)]^{1/2} \bullet \mathrm{d}B(t) \mid P(t)=p \bigr\rangle \;=\; \alpha'(p)\, M\, k_\mathrm{B}\mathcal{T}\, \mathrm{d}t.$$

Combining this with the conditional expectation for the friction term in Eq. (2.54),

$$\bigl\langle -\alpha(P)\, P\, \mathrm{d}t \mid P(t)=p \bigr\rangle \;=\; -\alpha(p)\, p\, \mathrm{d}t,$$

[13] We denote by $\langle \quad | P(t)=p\rangle$ the conditional expectation with respect to the Wiener measure of the Brownian motion $B(t)$.

we find for the lhs. of Eq. (2.57) the result

$$\left\langle \frac{\mathrm{d}P(t)}{\mathrm{d}t} \,\middle|\, P(t)=p \right\rangle = -[\alpha(p)\,p - \alpha'(p)\; M\; k_{\mathrm{B}}\mathcal{T}], \tag{2.58b}$$

Hence, by virtue of Eqs. (2.58), we see that the drift criterion (2.57) is equivalent to the following ordinary differential equation (ODE) for $\alpha(p)$:

$$-\alpha(p)\,p\; + \alpha'(p)\; M\; k_{\mathrm{B}}\mathcal{T} = g(p). \tag{2.59}$$

With respect to the two criteria formulated above, the solution of this ODE yields the friction function α that provides the 'best' Langevin approximation to the binary collision model. Information about the collision model and the bath distribution is encoded in the mean drift force $g(p)$. Evidently, the procedure leading to Eq. (2.59) can be generalized to other interaction models/bath distributions as well – provided the stationary distribution of the Brownian particle is known. Other types of interactions (e.g., nonelastic) would result in another function $g(p)$. A non-Maxwellian bath distribution would affect not only the rhs. of Eq. (2.59) but also its lhs. due to a modified fluctuation-dissipation relation.

In order to analyze Eq. (2.59) for our specific collision model, it is useful to define rescaled dimensionless quantities

$$p_* := p/p_{\mathrm{B}}, \qquad \mu_* := m/M, \qquad \alpha_*(p_*) := \alpha(p)\;/(2\,n_{\mathrm{b}}k_{\mathrm{B}}\mathcal{T}/p_{\mathrm{B}}),$$

allowing us to rewrite Eq. (2.59) in the form

$$\alpha_*(p_*)\;p_* - \frac{\mu_*}{2}\;\alpha_*'(p_*) \;\;=\;\; \frac{p_*}{\pi^{1/2}}\exp(-p_*^2) + \left(p_*^2+\frac{1}{2}\right)\;\mathrm{erf}(p_*). \tag{2.60}$$

This linear inhomogeneous first order ODE can be solved by standard numerical methods, but one still needs to specify an initial condition, e.g., the value $\alpha_*(0)$. The correct choice of $\alpha_*(0)$ can, in principle, be determined from the expected asymptotic behavior of $\alpha_*(p_*)$ at large momentum values $|p_*| \to \infty$. Considering the limit $|p_*| \to \infty$, we find that the solutions of Eq. (2.60) must converge to the solutions of the following asymptotic ODE

$$\tilde{\alpha}_*(p_*)\;p_* - \frac{\mu_*}{2}\;\tilde{\alpha}_*'(p_*) \;\;=\;\; \left(p_*^2+\frac{1}{2}\right)\mathrm{sign}(p_*). \tag{2.61}$$

The general solution of Eq. (2.61) with initial value $\tilde{\alpha}_*(0)$ reads

$$\tilde{\alpha}_*(p_*) = |p_*| + e^{p_*^2/\mu_*}\left[\tilde{\alpha}_*(0) - \frac{\pi^{1/2}}{2}\left(\mu_*^{1/2}+\mu_*^{-1/2}\right)\;\mathrm{erf}\big(\mu_*^{-1/2}|p_*|\big)\right]. \tag{2.62}$$

The asymptotic solution (2.62) implies that the friction coefficient α would grow or decrease exponentially unless one chooses a particular initial condition. Hence, the physically correct

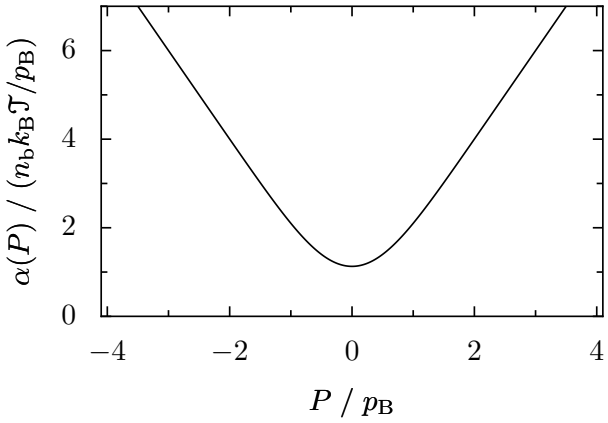


Figure 2.2: Nonrelativistic binary collision model. Nonlinear friction coefficent $\alpha(P)$ from Eq. (2.64), with $p_\mathrm{B} = M(2k_\mathrm{B}\mathcal{T}/m)^{1/2}$ denoting the characteristic thermal momentum of the Brownian particle (mass M) and $n_\mathrm{b} = N/L$ the number density of the heat bath particles (mass m).

asymptotic behavior, which is given by $\alpha_*(p_*) \simeq |p_*|$ for $p_* \to \pm\infty$, suffices to single out the specific initial condition that had to be used in the general solution of Eq. (2.59). Unfortunately, it seems to be very difficult or perhaps even impossible to find the exact analytical solution of the ODE (2.59).

Therefore, for practical purposes, one could use the simpler *ad hoc* approximation

$$\alpha(P) \;\approx\; \frac{2}{\delta t} \sum_{r=1}^{N} \frac{m}{M}\; \langle I_r(t,\delta t)\rangle_\mathrm{b}\,, \tag{2.63}$$

which reflects the earlier heuristic interpretation of Eq. (2.48) in terms of 'friction' and 'noise' contributions. One then finds

$$\alpha(P) \;\approx\; 2\,n_\mathrm{b} \frac{k_\mathrm{B}\mathcal{T}}{p_\mathrm{B}} \left\{ \pi^{-1/2} \exp\left[-\left(\frac{P}{p_\mathrm{B}}\right)^2 \right] + \left(\frac{P}{p_\mathrm{B}}\right)\; \mathrm{erf}\left(\frac{P}{p_\mathrm{B}}\right) \right\}. \tag{2.64}$$

Figure 2.2 depicts the nonlinear friction coefficient function $\alpha(P)$ from Eq. (2.64). A Langevin equation based on $\alpha(P)$ from Eq. (2.64) and $D(P) = \alpha(P) M k_\mathrm{B}\mathcal{T}$ yields the correct stationary momentum distribution (2.55), but does not exhibit exactly the same mean relaxation behavior as Eq. (2.48). Nonetheless, even such an approximate Langevin equation will provide a considerably more accurate description of the Brownian motion in a gaseous heat bath than, e.g., the classical Ornstein-Uhlenbeck process, which assumed constant friction and noise coefficients, cf. Eq. (2.3b). For instance, an Ornstein-Uhlenbeck (or Stokes-like) approximation could be obtained by replacing $\alpha(P)$ through its minimum

value

$$\alpha(0) = n_{\mathrm{b}} \frac{m}{M} \left(\frac{2k_{\mathrm{B}}\mathcal{T}}{\pi m} \right)^{1/2}, \tag{2.65a}$$

$$D(0) = \alpha(0)\, M\, k_{\mathrm{B}}\mathcal{T}. \tag{2.65b}$$

Adopting this additional simplification, the corresponding Ornstein-Uhlenbeck process can be expected to provide a useful description for slow Brownian particles at sufficiently low temperature values $\mathcal{T}$.

Although not quite as rigorous as the derivation from the harmonic oscillator model in Section 2.2.1, the collision model provides useful insights into the approximations that must be made in order to obtain a Langevin equation from a microscopic model. Compared with the oscillator model, a main advantage of the collision model is given by the fact that it can be extended to special relativity, cf. discussion in Section 4.3.

Chapter 3

Relativistic equilibrium thermostatistics

The preceding discussion of the nonrelativistic Brownian motion theory has shown that equilibrium thermostatistics plays an important role in constraining the relation between friction and noise coefficients in Langevin equations. Therefore, the present chapter intends to clarify several aspects of relativistic equilibrium thermostatistics. These considerations will become relevant in Chapter 4, when we will have to specify the fluctuation-dissipation relations for the relativistic Langevin theory.

Evidently, the nonrelativistic Brownian motion models from Chapter 2 are in conflict with special relativity because they do not prevent particles from moving faster than the speed of light c. For example, the stationary velocity distribution of the classical, e.g., Ornstein-Uhlenbeck process (2.1) is given by a Maxwell distribution that is non-zero for velocities $|\boldsymbol{v}| > c$. Hence, if one wants to construct a relativistic version of the Ornstein-Uhlenbeck process then one has to know the relativistic generalization of the Maxwell distribution first. The recent literature has seen considerable debate about the correct generalization of Maxwell's velocity distribution in special relativity [14, 16, 203, 211, 215, 216]. In Sec. 3.2 we shall present results of fully relativistic 1D molecular dynamics simulations [17] which clearly favor a distribution that was proposed by Jüttner [167] in 1911, i.e., six years after Einstein had formulated his theory of special relativity [2, 3]. Moreover, as discussed in the last part of this chapter, our simulations can also be used to illustrate the meaning of temperature and thermal equilibrium in special relativity [17].

3.1 Preliminaries

Section 3.1.1 contains a brief summary of the definitions and the notation that will be used in the remainder. Subsequently, the 'peculiar' transformation behavior of one-particle phase space PDFs under Lorentz transformations will be addressed. For simplicity, we restrict ourselves to discussing the 1D case (i.e., one time dimension and one space dimension); the corresponding generalization to higher space dimensions is straightforward.

3.1.1 Notation and conventions

The position of point-like particle in the inertial lab frame Σ at *lab time* t is denoted by $X(t)$; its lab velocity is defined by $V(t) := \mathrm{d}X(t)/\mathrm{d}t$. The relativistic momentum $P(t)$ and the relativistic particle energy $E(t)$ with respect to Σ are given by

$$P(t) := c\,M\,V(t)\,\gamma(V(t))\,, \qquad E(t) := c^2 M\,\gamma(V(t)), \tag{3.1}$$

with $M > 0$ denoting the particle rest mass and Lorentz factor

$$\gamma(v) := \left(1 - v^2/c^2\right)^{-1/2}. \tag{3.2}$$

To keep the subsequent formulas as simple as possible, we will from now on adopt a natural unit system with $c = 1$ yielding, e.g., the simplified relations

$$P = EV = MV\gamma(V)\,, \qquad E = (M^2 + P^2)^{1/2} = M\gamma(V). \tag{3.3}$$

Time and position, and energy and momentum can be combined into the contravariant four-vectors[1]

$$(X^\alpha) := (t, X), \qquad (P^\alpha) := (E, P), \qquad \alpha = 0, 1. \tag{3.4}$$

The components X_α and P_α of the corresponding covariant four-vectors are defined by

$$X_\alpha := \eta_{\alpha\beta} X^\beta, \qquad P_\alpha := \eta_{\alpha\beta} P^\beta, \qquad (\eta_{\alpha\beta}) = \mathrm{diag}(-1, 1). \tag{3.5}$$

Here, $\eta_{\alpha\beta}$ denote the components of the Minkowski metric tensor, and Einstein's summation convention has been used, i.e.,

$$\eta_{\alpha\beta} X^\beta := \sum_\beta \eta_{\alpha\beta} X^\beta.$$

The above definitions refer to the inertial lab frame Σ. Now consider a second inertial frame Σ', moving with velocity w relative to Σ. According to Einstein's theory of special

[1] We shall use the term 'four-vector' regardless of the actual number of spatial dimensions. Contravariant (covariant) four-vectors will be treated as column (row) vectors.

relativity, values of physical quantities in Σ' can be related to those in Σ by means of a Lorentz transformation [6, 8]. In the 1D case, the corresponding Lorentz transformation matrix reads

$$\Lambda(w) = \gamma(w) \begin{pmatrix} 1 & -w \\ -w & 1 \end{pmatrix} = (\Lambda(w)^{\alpha}{}_{\beta}). \tag{3.6}$$

The inverse $\Lambda^{-1}(w)$ is equal to $\Lambda(-w)$. The transformation law of an arbitrary four-vector (A^{α}) reads $A'^{\alpha} = \Lambda(w)^{\alpha}{}_{\beta} A^{\beta}$, leading in the case of (X^{α}) and (P^{α}) to the explicit results

$$\begin{pmatrix} t' \\ X' \end{pmatrix} = \gamma(w) \begin{pmatrix} t - wX \\ -wt + X \end{pmatrix}, \qquad \begin{pmatrix} E' \\ P' \end{pmatrix} = \gamma(w) \begin{pmatrix} E - wP \\ -wE + P \end{pmatrix}. \tag{3.7}$$

The rest mass M of a particle is Lorentz invariant, i.e.,

$$M = (E^2 - P^2)^{1/2} = (-P_{\alpha} P^{\alpha})^{1/2} = (-P'_{\alpha} P'^{\alpha})^{1/2} = (E'^2 - P'^2)^{1/2} = M'.$$

Throughout, we will assume that the rest mass of a particle is *not* changed by interactions. Finally, considering a particle with velocity curve $V(t)$, its Lorentz invariant proper time span $\Delta\tau$, elapsing between coordinate times t_1 and t_2, is given by

$$\Delta\tau := \int_{t_1}^{t_2} \mathrm{d}t \, [1 - V(t)^2]^{1/2} \qquad \Leftrightarrow \qquad \mathrm{d}\tau := \mathrm{d}t \, [1 - V(t)^2]^{1/2}. \tag{3.8}$$

3.1.2 Probability densities in special relativity

With regard to the subsequent discussion, it is worthwhile to clarify the definition and transformation behavior of PDFs in special relativity. The results presented in this subsection were proven rigorously by van Kampen [138].

To start with, we consider the *one-particle* phase space PDF $f(t, x, p) \geq 0$, where the coordinates (t, x, p) refer to the lab frame Σ with $p = m\gamma(v)v$ denoting the relativistic kinetic momentum. For a relativistic many-particle system (e.g., a gas) with conserved particle number $N \gg 1$, the function f can be defined operationally as follows [138]:

If an observer $\mathcal{O}$, who is at rest in Σ, observes the system at Σ-time t, he finds

$$N \, f(t, x, p) \, \mathrm{d}x\mathrm{d}p$$

particles in the phase space interval $[x, x + \mathrm{d}x] \times [p, p + \mathrm{d}p]$. Alternatively, when considering the random motion of a single Brownian particle in a fluctuating medium, the quantity $f(t, x, p)\mathrm{d}x\mathrm{d}p$ gives the probability of finding the Brownian particle at lab time t in $[x, x + \mathrm{d}x] \times [p, p + \mathrm{d}p]$. In either case, the function f is subject to the t-simultaneous normalization condition

$$1 = \int \mathrm{d}x\mathrm{d}p \; f(t, x, p). \tag{3.9}$$

Now consider a second observer $\mathcal{O}'$, moving with velocity $w \neq 0$ relative to Σ. The observer $\mathcal{O}'$ will measure another distribution $f'(t', x', p')$ and one is led to wonder how the two functions $f'(t', x', p')$ and $f(t, x, p)$ are related to each other. In the nonrelativistic theory, the change from one inertial system to another does not affect the time coordinate; hence, one can use the standard transformation laws for PDFs in that case [see, e.g., Eq. (2.17)]. By contrast, in the relativistic case the situation becomes more complicated, because now the definition of f and f' is based on an observer-dependent notion of simultaneity. Put differently, the measurements of $\mathcal{O}$ and $\mathcal{O}'$ refer to the two different hyperplanes "t=constant" and "t'=constant" in Minkowski space, respectively. Van Kampen [138] was able to prove that the one-particle phase space density f transforms as a Lorentz scalar, i.e.,[2]

$$f(t, x, p) = f'(t', x', p'), \tag{3.10}$$

where (t, x, p) and (t', x', p') are connected by a Lorentz transformation with velocity parameter w. Moreover, he showed that the function f' satisfies the t'-simultaneous normalization condition

$$1 = \int \mathrm{d}x' \mathrm{d}p' \; f(t', x', p'). \tag{3.11}$$

Van Kampen's proof [138] of Eq. (3.10) uses an assumption about the uniqueness of particle trajectories and a reparameterization of the particles trajectories in terms of their invariant proper times. As a consequence, Eq. (3.10) represents a generic kinematical result and applies to a broad class of interaction models.

We next summarize several useful implications of Eq. (3.10). For this purpose, we define the marginal densities

$$\phi(t, p) = \int \mathrm{d}x \, f(t, x, p), \qquad \phi'(t', p') = \int \mathrm{d}x' \, f'(t', x', p'), \tag{3.12a}$$

$$\varrho(t, x) = \int \mathrm{d}p \, f(t, x, p), \qquad \varrho'(t', x') = \int \mathrm{d}p' \, f'(t', x', p'). \tag{3.12b}$$

Then the following statements hold true [138]:

(i) *Free particles.* An unconfined system consisting of free, identical particles of rest mass m (i.e., no interactions, no external fields, no walls) is described by a time-independent marginal momentum distribution $\phi(p)$. In this case, one finds

$$(m^2 + p^2)^{1/2} \, \phi(p) = (m^2 + p'^2)^{1/2} \, \phi'(p'). \tag{3.13}$$

[2] One can find several insufficient 'proofs' of Eq. (3.10) in the literature, cf. the discussion in [138, 201]. In this context it is often claimed, erroneously, that the phase space element $\mathrm{d}x\mathrm{d}p$ is a Lorentz scalar; in Section 2 of their paper, Debbasch et al. [201] demonstrate in detail that this is not true in general.

(*ii*) *Ideal gas in a container.* Consider a spatially homogeneous gas enclosed in a box that rests in the lab frame Σ. Assume that the gas is in equilibrium and can be described by a time-independent one-particle phase space density

$$f(x,p) = \frac{\phi(p)}{V} I(x)$$

in Σ, where V is the rest volume of the box in Σ and $I(x)$ the indicator for the box, i.e., $I(x) = 1$ if x is within the box and $I(x) = 0$, otherwise. In this case, Eq. (3.10) implies that[3]

$$\frac{\phi(p)}{V} = \frac{\phi'(p')}{V'}, \tag{3.14}$$

where $V' = V/\gamma(w)$ is the Lorentz-contracted box volume in the moving frame Σ'. At first sight it is surprising that the presence of the box alters the transformation properties of the momentum distribution. However, this can be explained by the fact that the observations by $\mathcal{O}$ and $\mathcal{O}'$ are not synchronous, and that in the time between their observations some particles collide with the container walls.[4]

(*iii*) *Current-density vector.* The quantities

$$\varrho(t,x) = \int \mathrm{d}p\, f(t,x,p)\,, \qquad\qquad j(t,x) = \int \mathrm{d}p\, v\, f(t,x,p) \tag{3.15}$$

can be combined into a current-density four-vector $(j^\alpha) = (\varrho, j)$, since they transform as

$$\varrho'(t',x') = \gamma(w)\,\varrho(t,x) - \gamma(w)\,w\,j(t,x), \tag{3.16a}$$

$$j'(t',x') = -\gamma(w)\,w\,\varrho(t,x) + \gamma(w)\,j(t,x). \tag{3.16b}$$

Furthermore, it can be shown that j^α satisfies the continuity equation

$$\partial_\alpha j^\alpha = \frac{\partial \varrho}{\partial t} + \mathrm{div} j = 0. \tag{3.17}$$

Equation (3.10) and statements $(i) - (iii)$ remain valid in higher space dimensions $d > 1$ upon replacing $x \to \boldsymbol{x}$, $p \to \boldsymbol{p}$, $j \to \boldsymbol{j}$, etc. [138]. Moreover, Eq. (3.10) can be generalized to the case of N-particle phase space PDFs f_N, yielding for arbitrary space dimensions

$$f_N(t_1, \boldsymbol{x}_1, \boldsymbol{p}_1; \ldots; t_N, \boldsymbol{x}_N, \boldsymbol{p}_N) = f'_N(t'_1, \boldsymbol{x}'_1, \boldsymbol{p}'_1; \ldots; t'_N, \boldsymbol{x}'_N, \boldsymbol{p}'_N), \tag{3.18}$$

where for $n = 1, \ldots, N$ the coordinates $(t'_n, \boldsymbol{x}'_n, \boldsymbol{p}'_n)$ and $(t_n, \boldsymbol{x}_n, \boldsymbol{p}_n)$ are connected by a Lorentz-transformation, and f_N is the *multiple-time* probability density for lab observer $\mathcal{O}$ to observe particle 1 at time t_1 near $(\boldsymbol{x}_1, \boldsymbol{p}_1)$, *and* particle 2 at time t_2 near $(\boldsymbol{x}_2, \boldsymbol{p}_2)$, etc..

The above results clarify the transformation behavior of PDFs in special relativity, but they do not yet answer the question as to *which* PDF provides the correct description for a given physical system as, e.g., a relativistic gas in equilibrium. The latter problem will be addressed in the next part.

[3] This result was already derived by Dirac [453] in 1924.

[4] Cf. discussion in Section 6 of van Kampen's paper [138].

3.2 Thermostatistics of a relativistic gas

The recent literature has seen considerable debate about the correct generalization of Maxwell's velocity distribution in special relativity [14, 16, 203, 211, 215, 216]. Knowing the correct equilibrium velocity distributions of relativistic many-particle systems is essential for a proper interpretation of experiments in high energy and astrophysics [312, 314, 317, 338]. Examples include thermalization processes in heavy ion collision experiments [312, 314] and ultra-relativistic plasma beams [338], or the relativistic Sunyaev-Zel'dovich (SZ) effect [317], describing the distortion of the cosmic microwave background (CMB) radiation spectrum due to the interaction of CMB photons with hot electrons in clusters of galaxies [454–456]. The predicted strength of the spectral distortions and the cosmological parameters inferred from the SZ effect depend on the assumed velocity distribution of the relativistic electrons [317].

The equilibrium velocity distribution of a relativistic gas does also play an important role in the context of relativistic Brownian motion theory. As mentioned before, when postulating relativistic Langevin equations [10,11,18,334], this distribution must be known in advance in order to correctly specify the relativistic fluctuation-dissipation relation. Furthermore, the relativistic equilibrium velocity distribution is required as an input, if one wants to derive relativistic Langevin-type equations from a microscopic collision model.

3.2.1 Relative entropy, Haar measures and canonical velocity distributions

At the beginning of the last century it was commonly accepted that the one-particle velocity distribution of a dilute (quasi-ideal) gas in equilibrium is described by the Maxwellian (2.51)

$$\psi_{\mathrm{M}}(\boldsymbol{v};m,\beta,d)=\left(\frac{\beta m}{2\pi}\right)^{d/2}\exp\!\left(-\frac{\beta m\boldsymbol{v}^2}{2}\right), \tag{3.19}$$

where m is the rest mass of a gas particle, $\boldsymbol{v}\in\mathbb{R}^d$ the velocity, $\mathcal{T}=(k_{\mathrm{B}}\beta)^{-1}$ the temperature, and d the number of space dimension. After Einstein [2, 3] had formulated his theory of special relativity in 1905, Planck and others noted immediately that the distribution (3.19) is in conflict with the fundamental relativistic postulate that velocities cannot exceed the light speed c. A first solution to this problem was put forward by Ferencz Jüttner [167] in 1911. He proposed the following relativistic generalization of Maxwell's PDF [167]:

$$\psi_{\mathrm{J}}(\boldsymbol{v};m,\beta_{\mathrm{J}},d)=\frac{m^d}{\mathcal{Z}_{\mathrm{J}}}\gamma(\boldsymbol{v})^{2+d}\exp[-\beta_{\mathrm{J}}m\gamma(\boldsymbol{v})],\qquad |\boldsymbol{v}|<1, \tag{3.20a}$$

with $\mathcal{Z}_{\mathrm{J}}=\mathcal{Z}_{\mathrm{J}}(m,\beta_{\mathrm{J}},d)$ being the normalization constant, and $\psi_{\mathrm{J}}\equiv 0$ if $|\boldsymbol{v}|\geq 1$. Assuming a spatially homogeneous gas distribution, Eq. (3.20a) corresponds to the one-particle phase

space PDF[5]

$$f_{\rm J}(\boldsymbol{x},\boldsymbol{p}) = (\mathcal{Z}_{\rm J}V)^{-1}\exp[-\beta_{\rm J}E(\boldsymbol{p})], \tag{3.20b}$$

where V is the volume of the gas container, $E(m,\boldsymbol{p}) = (m^2{+}\boldsymbol{p}^2)^{1/2} = m\gamma(\boldsymbol{v})$ the relativistic particle energy, and $\boldsymbol{p} = m\boldsymbol{v}\gamma(\boldsymbol{v})$ the relativistic momentum; the corresponding marginal momentum PDF reads[6]

$$\phi_{\rm J}(\boldsymbol{p}) = \mathcal{Z}_{\rm J}^{-1}\exp[-\beta_{\rm J}E(\boldsymbol{p})]. \tag{3.20c}$$

Equations (3.20) refer to a lab frame Σ where the box, enclosing the gas, is at rest.

Jüttner's proposal (3.20) became widely accepted among theorists during the first three quarters of the last century [112,162,169,273,289] – although a rigorous microscopic derivation remained lacking due to the difficulty of formulating a relativistically consistent Hamilton mechanics of interacting particles [223, 224, 226, 228, 252]. Doubts about the Jüttner function $f_{\rm J}$ began to arise in the 1980s, when Horwitz et al. [215, 216] suggested a 'manifestly covariant' relativistic Boltzmann equation, whose stationary solution differs from Eq. (3.20) and, in particular, predicts a different mean energy-temperature relation in the ultra-relativistic limit [206]. Since then, partially conflicting results and proposals from other authors [14,16,203,207,211] have led to an increasing confusion as to which distribution actually represents the correct generalization of the Maxwellian (3.19). For example, another recently discussed alternative to Eq. (3.20), which also reduces to the Maxwell distribution in the nonrelativistic limit case, is the 'modified' Jüttner function [14,16,211]

$$\psi_{\rm MJ}(\boldsymbol{v};m,\beta_{\rm MJ},d) = \frac{m^d}{\mathcal{Z}_{\rm MJ}}\frac{\gamma(\boldsymbol{v})^{2+d}}{m\gamma(\boldsymbol{v})}\exp[-\beta_{\rm MJ}m\gamma(\boldsymbol{v})]. \tag{3.21a}$$

For a spatially homogeneous gas, Eq. (3.21a) corresponds to the phase space PDF

$$f_{\rm MJ}(\boldsymbol{x},\boldsymbol{p}) = V^{-1}\phi_{\rm MJ}(\boldsymbol{p}), \tag{3.21b}$$

with marginal momentum density

$$\phi_{\rm MJ}(\boldsymbol{p}) = \frac{(\mathcal{Z}_{\rm MJ})^{-1}}{E(\boldsymbol{p})}\exp[-\beta_{\rm MJ}E(\boldsymbol{p})], \tag{3.21c}$$

Compared with the Jüttner distribution (3.20) at the same parameter values $\beta_{\rm J} = \beta_{\rm MJ} \lesssim 1/m$, the modified PDFs (3.21) exhibits a significantly lower particle population in the high energy tail because of the additional $1/E$-prefactor.

[5]Here and below, it will be assumed that f vanishes outside the gas container.

[6]Jüttner [167] derived the distribution (3.20) from a maximum entropy principle; cf. Section 3.2.1. An alternative derivation based on the microscopic ensemble is given by Matolcsi et al. [198].

Maximum relative entropy principle In a recent paper [16] we have demonstrated that the distributions (3.20) and (3.21) can be obtained from a common maximum relative entropy principle by using different reference densities, respectively. To briefly illustrate this, we return to the 1D case[7] $d = 1$ and consider two positive density functions $\phi(p) > 0$ and $\rho(p) > 0$ on the 1D relativistic momentum space[8] $\mathcal{RM}_1 = (-\infty, \infty)$. The functions ϕ and ρ define two measures [457] μ_ϕ and μ_ρ, respectively, upon assigning to any subset $\mathcal{A} \subseteq \mathcal{RM}_1$ the numbers[9]

$$\mu_\phi(\mathcal{A}) := \int_{\mathcal{A}} \mathrm{d}p\ \phi(p)\ , \qquad\qquad \mu_\rho(\mathcal{A}) := \int_{\mathcal{A}} \mathrm{d}p\ \rho(p). \tag{3.22}$$

The relative entropy $\mathfrak{S}$ of ϕ with respect to ρ (or, equivalently, of μ_ϕ with respect to μ_ρ) is defined by [458–460]

$$\mathfrak{S}[\phi|\rho] := -\int \mathrm{d}p\ \phi(p)\ \ln\frac{\phi(p)}{\rho(p)}. \tag{3.23}$$

Here, as usual, the symbol $\int \mathrm{d}p$ signals an integration with respect to the Lebesgue measure λ on $\mathcal{RM}_1 = (-\infty, \infty)$, which assigns to each interval $\mathcal{A} = [a, b] \subseteq \mathbb{R}$ the intuitive measure number [457]

$$\lambda(\mathcal{A}) = \int_{\mathcal{A}} \mathrm{d}p = \int_a^b \mathrm{d}p = b - a.$$

It is worthwhile to note that the relative entropy (3.23) is manifestly invariant under coordinate transformations [16]. In the mathematical literature, the definition of the relative entropy is often given in the form

$$\hat{\mathfrak{S}}[\mu_\phi|\mu_\rho] := -\int \mathrm{d}\mu_\phi\ \ln\frac{\mathrm{d}\mu_\phi}{\mathrm{d}\mu_\rho}, \tag{3.24}$$

which upon identifying

$$\mathrm{d}\mu_\phi = \mathrm{d}p\ \phi(p)\ , \qquad\qquad \mathrm{d}\mu_\rho = \mathrm{d}p\ \rho(p)\ , \qquad\qquad \frac{\mathrm{d}\mu_\phi}{\mathrm{d}\mu_\rho} = \frac{\phi(p)}{\rho(p)}$$

becomes equivalent to Eq. (3.23). The relative entropy is also known as the Kullback-Leibler entropy [461], and the non-negative function $\frac{\mathrm{d}\mu_\phi}{\mathrm{d}\mu_\rho}(p)$ is the Radon-Nikodym density [457] of μ_ϕ with respect to μ_ρ. In the remainder, we are going to work with the density representation (3.23).

[7] See Ref. [16] for a discussion of the d-dimensional case.

[8] Since we restrict ourselves to the spatially homogeneous case, it suffices to focus on the momentum distribution ϕ.

[9] In particular, if $\mu_\phi(\mathcal{RM}_1) = 1$ holds true then $\phi \geq 0$ is a PDF and μ_ϕ a probability measure on $\mathcal{RM}_1$; at this point, however, we do not require normalization of ϕ or ρ.

The relative entropy $\mathfrak{S}[\phi|\rho]$ from Eq. (3.23) constitutes the basis of the maximum relative entropy principle. This principle formalizes the idea that, for a broad class of physical systems, the canonical equilibrium distribution can be obtained by maximizing an appropriate entropy functional in the presence of constraints. The constraints encode *a priori* knowledge about the system. In our case, we are interested in maximizing $\mathfrak{S}[\phi|\rho]$ with respect to ϕ under the conditions

$$1 = \int \mathrm{d}p\ \phi(p), \tag{3.25a}$$

$$\epsilon = \int \mathrm{d}p\ E(p)\,\phi(p). \tag{3.25b}$$

The first constraint (3.25a) ensures that ϕ is a PDF. The second constraint (3.25b) reflects the assumption that the mean energy per particle, ϵ, is known. These constraints may be incorporated into the maximum entropy principle via the method of Lagrangian multipliers [462]. Denoting the Lagrangian multipliers by (α, β), the maximum entropy principle results the necessary condition

$$\begin{aligned} 0 &\equiv \frac{\delta}{\delta\phi}\left\{\mathfrak{S}[\phi|\rho] + \alpha\left[1 - \int \mathrm{d}p\ \phi(p)\right] + \beta\left[\epsilon - \int \mathrm{d}p\ E(p)\,\phi(p)\right]\right\}\Big|_{\phi=\phi_*} \\ &= -\left[1 + \ln\frac{\phi_*(p)}{\rho(p)}\right] - \alpha - \beta E(p), \end{aligned} \tag{3.26}$$

whose solution is readily obtained as[10]

$$\phi_*(p) = \varrho(p)\ \exp\big[-(1+\alpha) - \beta E(p)\big]. \tag{3.27}$$

The Lagrangian multipliers (α, β) can be determined from the two conditions (3.25) yielding, e.g., β as a function of the given mean value ϵ. Moreover, it is evident now that normalization of the reference density ρ is irrelevant, since constant prefactors will be absorbed by the multiplier α. We next discuss three specific cases:

a) *Maxwell distribution.* Upon choosing a constant reference density in momentum space, $\rho(p) = \rho_0$, and the nonrelativistic kinetic energy $E(p) = p^2/(2m)$, the solution (3.27) yields the 1D nonrelativistic Maxwell momentum distribution (2.16).

b) *Jüttner distribution.* Fixing again a constant reference density, $\rho(p) \equiv \rho_0$, and considering the relativistic energy $E(p) = (m^2 + p^2)^{1/2}$, one recovers Jüttner's momentum PDF (3.20c).

[10] Due to the appearance of the logarithm, the maximization of the entropy functional (3.23) gives rise to an 'exponential' distribution (3.27). By considering other non-logarithmic 'entropies' like, e.g., Tsallis' or Kaniadakis' entropy, one may construct other forms of distributions (e.g., power law distributions).

c) *Modified Jüttner distribution.* Considering, as in b), the relativistic energy $E(p) = (m^2 + p^2)^{1/2}$, but now with a non-constant reference density $\rho(p) = E(p)^{-1}$ in Eq. (3.27), one obtains the modified Jüttner PDF (3.21c).

This shows that the two candidate distribution (3.20) and (3.21) may be derived from a common maximum relative entropy principle, but they refer to different reference densities (i.e., reference measures), respectively. It is worthwhile to clarify the difference between the two reference measures by analyzing their respective symmetries:

The constant reference density $\rho(p) \equiv \rho_0$, underlying the Jüttner distribution (3.20), corresponds to the Lebesgue measure λ on $\mathcal{RM}_1$, i.e., in this case we have

$$\mu_\rho(\mathcal{A}) = \rho_0 \, \lambda(\mathcal{A}) \, , \qquad \forall \, \mathcal{A} \in \mathcal{RM}_1 = \mathbb{R}. \tag{3.28}$$

The associated relative entropy

$$\mathfrak{S}[\phi|\rho_0] := -\int \mathrm{d}p \; \phi(p) \; \ln \frac{\phi(p)}{\rho_0} \tag{3.29}$$

coincides with the usual Shannon-Boltzmann-Gibbs entropy on $\mathcal{RM}_1$. The Lebesgue measure λ on $\mathcal{RM}_1$ is distinguished by the fact that it is the only translation invariant measure on the relativistic momentum space. Here, translation invariance of λ means that

$$\lambda([a+x, b+x]) = (b+x) - (a+x) = b - a = \lambda([a,b])$$

holds for all $a, b, x \in \mathcal{RM}_1$. In more mathematical terms, λ represents the Haar measure[11] of the (additive) momentum translation group $(\mathcal{RM}_1, +)$.

[11] In a seminal paper [463] published in 1933, the Hungarian mathematician Alfred Haar studied the possibility to introduce a measure $\mu_\circ$ *on* a continuous group $(\mathcal{G}, \circ)$ such that $\mu_\circ$ is invariant under the group multiplication '$\circ$'. To briefly sketch this idea, consider a subset $\mathcal{A}$ of the group $\mathcal{G}$ and some arbitrary, fixed group element $g \in \mathcal{G}$. By multiplying each element $a \in \mathcal{A}$ with g, the subset $\mathcal{A}$ is mapped onto another subset of $\mathcal{G}$, denoted by

$$g \circ \mathcal{A} := \{ g \circ a \, | a \in \mathcal{A} \} \, .$$

Now consider a measure $\mu_\circ$ on $\mathcal{G}$ that assigns to $\mathcal{A} \subseteq \mathcal{G}$ some non-negative real number $\mu_\circ(\mathcal{A})$. The measure $\mu_\circ$ is said to be group invariant, if

$$\mu_\circ(g \circ \mathcal{A}) = \mu_\circ(\mathcal{A})$$

holds for any $g \in \mathcal{G}$ and $\mathcal{A} \subseteq \mathcal{G}$. In the case of non-commutative (i.e., non-Abelian) groups, one may distinguish invariance under multiplications from the right or left. Haar [463] was able to prove the existence of an invariant measure $\mu_\circ$, and its uniqueness apart from an irrelevant multiplicative constant for locally compact, topological groups. Such group invariant measures $\mu_\circ$ are referred to as Haar measures nowadays [457]. They give a mathematically precise meaning to the notion 'uniform distribution' by combining measure and group theoretical concepts.

Now consider the reference density $\rho(p) = E(p)^{-1}$, which yields the modified momentum PDF (3.21c). We may define the Lorentz transformation $L[\mathcal{A}]$ of a set $\mathcal{A} \subset \mathcal{RM}_1$ by

$$L[\mathcal{A}] := \{L(p) \mid p \in \mathcal{A}\} . \tag{3.30}$$

By taking into account the well-known fact that [138, 290]

$$\frac{\mathrm{d}p}{E(p)} = \frac{\mathrm{d}p'}{E(p')} \tag{3.31}$$

holds under Lorentz transformations, one then finds that

$$\mu_\rho(L[\mathcal{A}]) = \mu_\rho(\mathcal{A}) , \qquad \forall\, \mathcal{A} \in \mathcal{RM}_1. \tag{3.32}$$

That is, the measure μ_ρ induced by $\rho(p) = E(p)^{-1}$ is invariant under Lorentz transformations.

To briefly summarize, adopting the Lebesgue measure in momentum space as reference measure yields the Jüttner function (3.20c), whereas the Lorentz invariant reference measure yields the modified distribution (3.21c). Hence, at this stage, both distribution functions appear to be plausible candidates,[12] and it remains the question which reference measure is the physically correct one. This question cannot be answered on the level of the maximum entropy principle and, therefore, other approaches have to be explored. To identify the physically correct distribution, we have performed numerical simulations of a fully relativistic dynamical 1D gas model that will be discussed in the next section.

3.2.2 Relativistic molecular dynamics simulations

To resolve the uncertainty about the relativistic one-particle equilibrium velocity PDF, we performed fully relativistic 1D molecular dynamics (MD) simulations in collaboration with David Cubero and Jesús Casado from the Universidad de Sevilla [17]. The restriction to the 1D case is inevitable if one wants to treat localized particle interactions in a relativistically consistent manner, cf. the remarks below. In our computer experiments we simulated the dynamics of classical, impenetrable point-particles with elastic point-like binary collisions, employing an algorithm similar to those of Alder and Wainwright [464] and Masoliver and Marro [465].

[12] As shown in [14], the modified distribution satisfies an additional elastic collision invariance criterion; this microscopic criterion, however, does not take into account the collision rates and is, therefore, not appropriate for determining the equilibrium distribution of a confined system as, e.g., a relativistic gas in a container.

The basic time step of the algorithm involves three partial tasks:

(*i*) determine the next collision event (x_c, t_c);

(*ii*) evolve the system up to time t_c;

(*iii*) calculate the momenta after the collision.

The third task is solved as follows: If two particles A and B meet at the space-time point (x_c, t_c), then they exchange momentum according to the relativistic energy momentum conservation laws

$$p_A + p_B = \hat{p}_A + \hat{p}_B, \tag{3.33a}$$

$$E(m_A, p_A) + E(m_B, p_B) = E(m_A, \hat{p}_A) + E(m_B, \hat{p}_B), \tag{3.33b}$$

where $E(m, p) = (m^2 + p^2)^{1/2}$ is the relativistic particle energy. Given the momenta (p_A, p_B) before the collision, these conservation laws determine the momenta $(\hat{p}_A, \hat{p}_B)$ after the collision by [14]

$$\hat{p}_A = \gamma(u)^2[2uE(m_A, p_A) - (1 + u^2)p_A], \tag{3.34a}$$

$$\hat{p}_B = \gamma(u)^2[2uE(m_B, p_B) - (1 + u^2)p_B], \tag{3.34b}$$

where

$$u = (p_A + p_B)/[E(m_A, p_A) + E(m_B, p_B)]$$

is the collision-invariant, relativistic center-of-mass velocity of the two particles. By assuming strictly localized, point-like pair interactions, one may avoid the introduction of fields which are required when considering relativistic particle interactions-at-a-distance.[13] However, considering point-like localized interactions is expedient in the 1D case only; in higher space dimension the collision probability would become zero, thus preventing the system from equilibration. Moreover, if two colliding particles carry the same rest masses then elastic 1D collisions merely interchange their velocities; hence, elastic binary collisions are not able to drive a 1D *one-component* gas to equilibrium. In our simulations we considered a *two-component* mixture, consisting of N_1 light particles with equal masses m_1, and N_2 heavy particles with equal masses $m_2 > m_1$. The motion of the $N = N_1 + N_2$ particles was restricted to the 1D interval $[0, L]$, assumed to be stationary in the lab frame Σ. The numerical results presented below refer to elastic reflections at the boundaries; however, we found that periodic boundary conditions yield identical outcomes if the total initial momentum was chosen to be zero in Σ. Generally, our simulations mimic a relativistic

[13] The interested reader may wish to consult the original papers of Wheeler and Feynman [223], Currie et al. [226], and Van Dam and Wigner [224, 225], who discuss in detail the difficulties associated with classical particle-particle interactions in special relativity.

microcanonical ensemble, since the total initial energy $E_{\rm tot}$ in Σ is conserved in the microscopic collision processes. The above conventions define the simplest interacting model system that

a) complies with all principles of special relativity,

b) does not require the introduction of interaction fields,

c) can be simulated without further approximation, and

d) exhibits a universal stationary equilibrium state.

Hence, this simple model is well-suited for probing the predictions of different relativistic kinetic theories [162, 169, 206, 215, 273, 289] by means of numerical experiments. Moreover, as we shall see below, this model may help to clarify longstanding questions regarding the definition and meaning of 'temperature' and 'thermal equilibrium' in special relativity.

Numerical results In order to identify the stationary one-particle velocity distributions for the light and heavy particles, respectively, we waited until the 1D two-component gas had approached the equilibrium state (typically, after 10^2 collisions per particle). Then the particle velocities were measured Σ-simultaneously, i.e., at equal time t with respect to the resting lab frame Σ. To increase the sample size we repeated this procedure several times during a simulation run and collected the data into a single histogram. An example is shown in Fig. 3.1, based on a simulation with $N = 2000$ particles ($N_1 = N_2 = 1000$, $m_2 = 2m_1$). Each particle was given a random initial position $x_i(0) \in [0, L]$ and a random initial velocity $v_i(0) = \pm 0.8$, corresponding to a mean energy per particle $\bar{\epsilon} = E_{\rm tot}/(N_1 + N_2) = 2.5m_1$. As evident from Fig. 3.1, for both particle species the numerically obtained one-particle PDFs ($\circ$) are in very good agreement with the standard Jüttner function $\psi_{\rm J}$ (solid line), and differ significantly from the modified distribution $\psi_{\rm MJ}$ (dashed lines).

The distribution parameters $\beta_{\rm J/MJ}$ underlying the graphs in Fig. 3.1 were determined from the initial energy by means of the following consideration: If the particle numbers N_1 and N_2 are sufficiently large (thermodynamic limit), then the one-particle PDFs in the lab frame Σ are expected to converge to either $\psi_{\rm J}$ from Eq. (3.20a) or $\psi_{\rm MJ}$ from Eq. (3.21a). Generally, the mean relativistic energy value ϵ of a one-particle velocity PDF $\psi(\boldsymbol{v}; m, \beta)$ is defined by

$$\epsilon(m, \beta) = \int_{\{|\boldsymbol{v}|<1\}} \mathrm{d}^d\boldsymbol{v}\ \psi(\boldsymbol{v}; m, \beta)\ m\gamma(\boldsymbol{v}). \tag{3.35}$$

Assuming that (i) an equilibrium state exists where both species can be described by the same value β, and that (ii) for a gas in equilibrium the mean energy per particle is the same for particles *of the same species*, the total energy can be expressed as

$$E_{\rm tot} = N_1\ \epsilon(m_1, \beta) + N_2\ \epsilon(m_2, \beta). \tag{3.36}$$

In our case, the energy mean values $\epsilon_{\mathrm{J/MJ}}$ of the two 1D candidate PDFs ψ_{J} and ψ_{MJ} can be calculated analytically, yielding [cf. App. B.1]

$$\epsilon_{\mathrm{J}}(m, \beta_{\mathrm{J}}) = m\frac{K_0(\beta_{\mathrm{J}}m) + K_2(\beta_{\mathrm{J}}m)}{2K_1(\beta_{\mathrm{J}}m)}, \tag{3.37a}$$

$$\epsilon_{\mathrm{MJ}}(m, \beta_{\mathrm{MJ}}) = m\frac{K_1(\beta_{\mathrm{MJ}}m)}{K_0(\beta_{\mathrm{MJ}}m)}, \tag{3.37b}$$

with K_n denoting modified Bessel functions of the second kind [466]. For each simulation run the parameter tuple $(E_{\mathrm{tot}}, N_1, N_2, m_1, m_2)$ is known. Hence, upon inserting them into

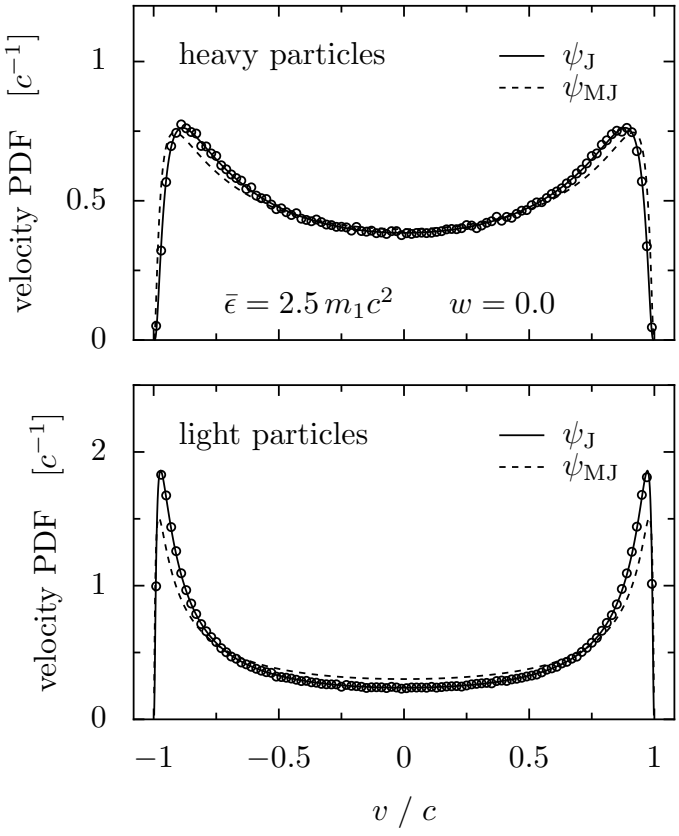


Figure 3.1: Equilibrium velocity PDFs in the lab frame Σ: Numerically obtained one-particle velocity PDFs ($\circ$) based on a simulation with $N_1 = 1000$ light particles of mass m_1 and $N_2 = 1000$ heavy particles with mass $m_2 = 2m_1$. The mean energy per particle is $\bar{\epsilon} = E_{\mathrm{tot}}/(N_1 + N_2) = 2.5m_1c^2$. The solid curves in the upper and lower panel correspond to Jüttner functions (3.20a) with same inverse temperature parameter $\beta_{\mathrm{J}} = 0.701\,(m_1c^2)^{-1}$, but different particle masses, respectively. Dashed lines show the corresponding modified distribution (3.21a) with $\beta_{\mathrm{MJ}} = 0.402\,(m_1c^2)^{-1}$. The simulation data is consistent with the standard Jüttner distribution ψ_{J} (solid lines), and thus provides evidence against the modified distribution (3.21).

Eqs. (3.36) and (3.37), these parameters uniquely determine the parameter value $\beta_{\mathrm{J/MJ}}$ that is consistent with the chosen velocity PDF $f_{\mathrm{J/MJ}}$.

Temperature and thermal equilibrium in special relativity Remarkably, in spite of the different particle masses the two numerically obtained velocity PDFs in Fig. 3.1 are very well matched by Jüttner functions (3.20) with the *same* parameter β_{J}. According to our simulations, this holds true with high accuracy for a wide range of initial conditions and mass ratios. Hence, the Jüttner function does not only provide the best 'fit' to the numerical data, it also yields a well-defined concept of 'temperature' in special relativity: Intuitively, the temperature $\mathcal{T}$ is thought to be an intensive quantity that equilibrates to a common value if two or more systems are brought into contact with each other (i.e., may exchange different forms of energy). In our case, it is natural to consider the particle species as two different subsystems that may exchange energy via elastic collision processes. After a certain relaxation time, the combined system approaches a 'thermodynamic equilibrium state', where each subsystem is described by the same asymptotic, two-parametric velocity PDF $\psi_{\mathrm{J}}(v; m_i, \beta_{\mathrm{J}})$, differing only via the rest masses m_i. The commonly shared distribution parameter β_{J} may thus be used to *define* a relativistic equilibrium temperature by

$$\mathcal{T} := (k_{\mathrm{B}}\beta_{\mathrm{J}})^{-1}. \tag{3.38}$$

However, for this concept to be meaningful, a restriction of the accessible spatial volume is required – be it by means of periodic boundary conditions, or by imposing reflecting walls.[14] Otherwise, it cannot be expected that a many-particle system approaches a universal stationary state which is independent of the specific initial conditions. This observation has an important implication: Any (relativistic or non-relativistic) Boltzmann-type equation [206, 215, 216, 273, 289, 290, 292] that gives rise to a universal stationary velocity PDF implicitly assumes the presence of a spatial confinement, thus singling out a preferred frame of reference.

[14]The critical role of the boundary conditions in relativistic systems has been emphasized by Sinyukov [467] and van Kampen [137,138]. Loosely speaking, if a many-particle system has reached a universal stationary equilibrium state, then each particle 'knows' about the presence of the confinement/walls because equilibration typically requires momentum reversal at the walls, e.g., in order to maintain a uniform density and a well-balanced average collision frequency. In particular, if the walls are considered to be stationary objects then they single out a preferred frame of reference. The relevance of the boundary conditions is even more obvious in quantum mechanics/statistics due to their direct effect on the energy spectra and, thus, on the density of states. In fact, quantum mechanical arguments [170] suggest $\rho(p) \equiv \rho_0$ in Eq. (3.23).

Moving observers From our simulations we may further determine the equilibrium velocity distributions as seen from another frame Σ' moving with velocity w relative to the lab frame Σ. Figure 3.2 depicts results for $w = 0.2$ and same simulation parameters as in Fig. 3.1. In contrast to Fig. 3.1, the numerical data points in Fig. 3.2 were obtained by measuring velocities Σ'-simultaneously. The solid curves in Fig. 3.2 correspond to the PDF

$$\psi'_{\mathrm{J}}(v'; m, \beta_{\mathrm{J}}, w) = \frac{m\,\gamma(v')^3}{\mathcal{Z}_{\mathrm{J}}\,\gamma(w)} \exp[-\beta_{\mathrm{J}}\gamma(w)m\gamma(v')\,(1 + wv')] \tag{3.39}$$

[v' is the particle velocity in the moving frame Σ']. The velocity PDF (3.39) is obtained by making use of Eq. (3.14). For $w = 0$, the PDF ψ'_{J} reduces to the Jüttner function (3.20). Due to the excellent agreement between the numerical simulations and Eq. (3.39), we may state more precisely: Two relativistic gas components are in 'thermodynamic equilibrium' for any observer if their one-particle velocity PDFs are given by generalized Jüttner functions (3.39) with same parameters β_{J} *and* w. Only in this case the net energy transfer between the different gas components in the container vanishes. Last but not least, the above results shed light on a longstanding, highly debated question [127, 129–131, 158] originally posed by Landsberg [126, 128]:

Does a moving body appear cool? Evidently, the answer depends on the thermometers employed by different observers. Adopting, for the reasons discussed above, $\mathcal{T} := (k_{\mathrm{B}}\beta_{\mathrm{J}})^{-1}$ as a reasonable temperature definition, a moving observer with rest frame Σ' can measure $\mathcal{T}$ by exploiting the Lorentz invariant equipartition theorem[15] [128]

$$k_{\mathrm{B}}\mathcal{T} = m\gamma(w)^3 \left\langle \gamma(v')\,(v' + w)^2 \right\rangle', \tag{3.40a}$$

where

$$w = -\langle v' \rangle' \tag{3.40b}$$

is the mean velocity of the gas measured by the moving observer, and the averages $\langle\,\cdot\,\rangle'$ are taken Σ'-simultaneously. We verified the validity of Eq. (3.40a) explicitly by using simulation data obtained for different values of w, see Fig. 3.3. Hence, Eq. (3.40a) defines a Lorentz invariant statistical gas thermometer. Put differently, this intrinsic statistical thermometer determines the proper temperature of the gas by making use of simultaneously measured particle velocities only; thus, by adopting the statistical thermometer definition (3.40a), *moving bodies appear neither hotter nor colder*.[16]

[15] Equation (3.40a) is obtained by combining the microcanonical equipartition theorem for a Hamiltonian $H = \sum_{i=1}^{N_1} E(m_1, p_i) + \sum_{j=1}^{N_2} E(m_2, p_j)$ with the Lorentz invariance of the relativistic phase space PDF f.

[16] The mean value from Eq. (3.40a) can be used to measure the rest temperature, which plays a central role in van Kampen's [137] approach to relativistic thermodynamics. Evidently, upon multiplying

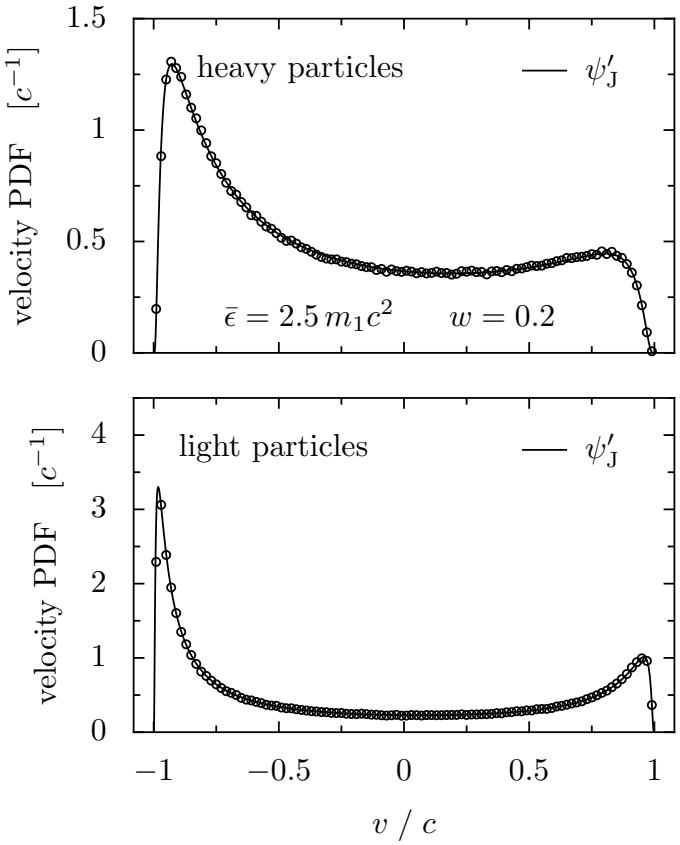


Figure 3.2: Equilibrium velocity PDFs in a moving frame Σ': Velocity PDFs as measured by an observer who moves with velocity $w = 0.2c$ relative to the lab frame Σ. Parameter values and initial conditions are the same as those in Fig. 3.1. The solid lines correspond to Jüttner functions ψ'_{J} from Eq. (3.39) with the same inverse temperature parameter $\beta_{\mathrm{J}} = 0.701\,(m_1 c^2)^{-1}$ as in Fig. 3.1 and different masses m_1 and m_2, respectively.

To briefly summarize the results of this part: Our fully relativistic MD simulations confirm the Jüttner distribution (3.20) as the correct relativistic one-particle equilibrium velocity distribution. Furthermore, our simulations corroborate van Kampen's [137] and Landsberg's view [126,128] that the temperature of classical gaseous systems *can* be defined and measured in a Lorentz invariant way.

The extension of the MD approach to higher space dimensions is nontrivial, due to the aforementioned difficulties of treating 2D and 3D two-body collisions in a relativistically consistent manner [223,224,226,228,252]. In order to be fully consistent with the require-

Eq. (3.40a) by factors $\gamma(w)^\alpha$, $\alpha \neq 0$ one can construct thermometers that measure 'other' temperatures; e.g., $a = -1$ would correspond to Planck's [111] formulation of relativistic thermodynamics and $a = 1$ to proposals made by Eddington [468] and Ott [115].

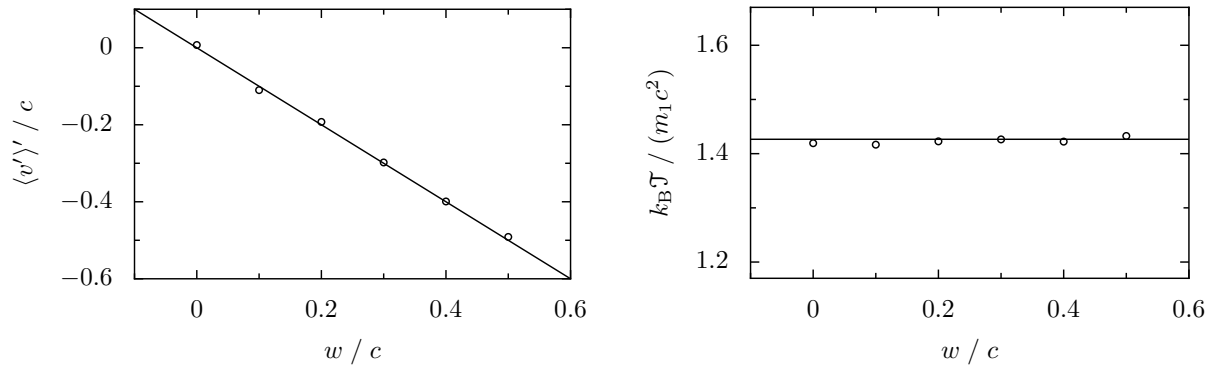


Figure 3.3: Measured mean particle velocity (∘, left digram) and estimated temperature (∘, right diagram) based on Eqs. (3.40) as a function of the observer velocity w. Solid lines correspond to the theoretically expected values, respectively, using the same simulation parameters and initial conditions as in Fig. 3.2.

ments of special relativity, relativistic 2D/3D interactions must be formulated in terms of fields. Unfortunately, direct simulation of the field dynamics is numerically expensive and, therefore, practically unfeasible in most cases. Alternatively, one can use simplified semi-relativistic models such as, e.g., effective hard-sphere models where the interaction radius is defined with respect the center-of-mass frame of the colliding particles [469]. Generally, it can be expected that such simplified models yield satisfactory results in the low density regime, but they may lead to inconsistencies at high densities, e.g., when three-body encounters become relevant.

We conclude this part with another general remark: The above results correspond the case where an 'ordinary' thermal equilibrium state is approached, corresponding to a stationary exponential one-particle momentum distributions. For completeness we note that, in general, one can also imagine non-equilibrium scenarios that give arise to (quasi-)stationary distributions which differ from the Jüttner function;[17] see, e.g., Kaniadakis [203, 208, 209], Silva and Lima [207, 210], Lina et al. [213], or Beck [204]. However, our subsequent discussion of relativistic Langevin equations will focus on the case, where the heat bath, which surrounds the Brownian particle, is described by a spatially homogeneous Jüttner function.

[17] Typical examples are unconfined systems with a limited number of collisions per particle as, e.g., in an expanding dilute gas.

Chapter 4

Relativistic Brownian motion

The present chapter discusses the generalization of the Langevin theory of Brownian motions to the framework of special relativity [10, 11, 13, 14, 16, 18–24, 331, 333–335, 374, 470]. More precisely, we will consider stochastic differential equations (SDEs) that describe Markov processes in relativistic one-particle phase space.[1] Relativistic Langevin equations present a useful tool for modeling the dynamics of relativistic particles in a random environment. Recently, for example, SDEs have been applied to analyze thermalization effects in quark-gluon plasmas produced at the Relativistic Heavy Ion Collider (RHIC) [312–314, 337].[2] The subsequent sections intend to give a comprehensive discussion and illustration of the underlying mathematical theory.

The phenomenological or axiomatic Langevin approach to relativistic Brownian motion was initiated by Debbasch et al. [18], who in 1997 proposed a simple relativistic generalization of the classical Ornstein-Uhlenbeck process [36]. The relativistic Ornstein-Uhlenbeck process (ROUP) of Debbasch et al. [18] is obtained by postulating additive white noise for the particle's momentum change in the rest frame of the bath. During the past decade various properties of the ROUP were studied by Debbasch and Rivet [19, 20], Barbachoux et al. [21,22], and Zygadło [334]. An alternative approach to relativistic Langevin equations was pursued by us in Refs. [10, 11]. By starting from a 'Newtonian' Ornstein-Uhlenbeck-type Langevin equation in the comoving rest frame of a Brownian particle, we obtained a modified relativistic Brownian motion (RBM) process, whose relaxation behavior differs from that of the ROUP. As we are going to illustrate below, the two different processes represent special limit cases of a larger class of relativistic Langevin models [12], which may be used to describe the random motions of relativistic particles.

The content of the present chapter is structured as follows. Section 4.1 discusses the axiomatic Langevin approach, i.e., suitable SDEs and fluctuation-dissipation relations are

[1] As discussed by Łopuszańnski [339], Dudley (Theorem 11.3 in [341]) and Hakim (Proposition 2 in [346]), it is impossible to define nontrivial relativistic Markov processes in position space.

[2] Potential applications in high energy astrophysics are discussed by Dieckmann et al. [338] and [471].

postulated in order to provide a simplified model of the complex interaction between the Brownian particle and its environment (heat bath). After having outlined the general conceptual foundations (Section 4.1.1), specific examples are considered (Section 4.1.2). In Section 4.1.3, we will analyze the temperature dependence of the asymptotic mean square displacement for different example processes, using general analytic formulas recently derived by Lindner [24], and Angst and Franchi [23]. In particular, we are going to demonstrate that the diffusion constant is sensitive with respect to variations of the friction coefficients. This result implies that measurements of the diffusion constant may reveal information about the underlying microscopic interactions. Section 4.2 discusses relativistic Brownian motion processes from the viewpoint of a moving observer. In the last part of this chapter (Section 4.3), we will generalize the binary collision model from Section 2.2.2 to the relativistic case in order to obtain a simple microscopic model for relativistic Brownian motions.

4.1 Langevin and Fokker-Planck equations

Roughly speaking, a relativistic Brownian motion process is a stochastic process whose absolute velocity $|\boldsymbol{V}(t)|$ does not exceed the speed of light c at any time. Of particular interest for our subsequent discussion are stochastic processes which

(i) satisfy the condition $|\boldsymbol{V}(t)| < c$, and

(ii) can be modeled by means of Langevin-type SDEs, or, equivalently, by means of Fokker-Planck equations (FPEs).

When considering Langevin equations as models of Brownian motion, one implicitly assumes that it is possible and reasonable to separate the degrees of freedom of the Brownian particle from those of the environment (heat bath). Adopting this point of view, one can specify two distinguished frames of reference: the stationary inertial rest frame Σ of the heat bath, and the inertial frame Σ_* that is comoving with the Brownian particle at a given instant of time.[3] As before, Σ is referred to as lab frame. In the present section, we focus on describing relativistic Brownian motion processes with respect to the space-time coordinates of Σ.

[3]Apart from an irrelevant shift of the origin, the inertial lab frame Σ is uniquely determined by the requirement that the mean velocity of the heat bath particles, which is assumed to be constant in any inertial frame, must vanish in Σ. Similarly, the instantaneously comoving frame Σ_* is determined by the condition that the Σ_*-velocity of the Brownian particle is equal to zero at the given instant of time. Generally, we assume that the time coordinates t and t_* can be measured, e.g., by using atomic clocks that are at rest in Σ or Σ_*, respectively.

4.1.1 Construction principle and conceptual aspects

The basic idea for constructing stochastic processes that meet the criteria (i) and (ii) is to postulate Langevin equations for the relativistic momentum coordinates $\boldsymbol{P} = (P^i)$, $i = 1, \ldots, d$ which can take values in $\mathbb{R} = (-\infty, +\infty)$. This automatically prevents particle velocities from exceeding the speed of light, because the associated absolute velocity

$$|\boldsymbol{V}| = \frac{|\boldsymbol{P}|}{P^0} = \frac{|\boldsymbol{P}|}{(M^2 + \boldsymbol{P}^2)^{1/2}}$$

is always less than $c = 1$ (throughout, $M > 0$ denotes the rest mass of the Brownian particle). However, before we can actually write down specific Langevin equations for the relativistic momentum components P^i, an important question needs to be addressed, namely, the choice of the time parameter in relativistic Langevin equations.

Choice of the time parameter A fundamental assumption (postulate) of nonrelativistic Galilean physics is the existence of a universal time t. Hence, within the nonrelativistic Langevin theory, it seems very natural to identify this universal time t with the time parameter of the stochastic driving process, often taken to be a multi-dimensional Wiener process $\boldsymbol{B}(t)$, cf. Eq. (2.3b). By contrast, in special relativity the notion of time is frame-dependent. Consequently, it becomes important to specify in advance which time parameter is used to quantify the fluctuations of the underlying stochastic driving process.

When considering the stochastic motion of a relativistic Brownian particle, two distinguished time parameters exist: The coordinate time t of the inertial lab frame Σ, defined as the mean rest frame of the stationary heat bath, and the proper time τ of the Brownian particle. In principle, either of the two parameters could be used to formulate SDEs for the spatial components of the particle momentum, $\boldsymbol{P} = (P^i)$. However, within the conventional Langevin picture of Brownian motion, one usually considers friction and noise as externally imposed forces that act upon the Brownian particle. Therefore, it seems more natural to characterize the statistical properties of the noise source in terms of the lab time t.[4]

Accordingly, within this lab time approach [10, 11, 18, 19, 21, 22, 333, 334, 472], one aims primarily at constructing $2d$-dimensional relativistic stochastic processes $\{\boldsymbol{X}(t), \boldsymbol{P}(t)\}$ with respect to the lab frame Σ, where the position coordinates $\boldsymbol{X} = (X^i)$ and the spatial momentum coordinates $\boldsymbol{P} = (P^i)$ are connected by the standard relativistic differential

[4]Within this lab time approach, the proper time becomes a stochastic quantity, and one could, for example, ask for the probability of finding at lab time t the particle's proper time in the interval $[\tau, \tau + \mathrm{d}\tau]$. Conversely, if adopting the proper time τ as the primary deterministic evolution parameter, one could ask for the probability to find the particle at proper time τ within the space-time interval $[t, t + \mathrm{d}t] \times [\boldsymbol{x}, \boldsymbol{x} + \mathrm{d}\boldsymbol{x}]$ with respect to the lab frame.

relation

$$\mathrm{d}X^i(t) = V^i\,\mathrm{d}t = (P^i/P^0)\,\mathrm{d}t, \qquad i = 1,\ldots,d, \tag{4.1}$$

with $P^0(t) = (M^2 + \boldsymbol{P}^2)^{1/2}$ denoting the relativistic energy. Stochasticity is implemented into the dynamics by coupling the momentum components $P^i(t)$ to an external noise source via an SDE. Below, we shall focus on the case where the noise source is modeled by a standard Wiener process. Analogous to the nonrelativistic case, the relativistic Langevin equation (RLE) for the stochastic increments $\mathrm{d}P^i(t)$ can be written using different kinds of discretization rules. In the following, we will consider the three most popular cases, corresponding to the post-point ($\bullet$), mid-point ($\circ$) and pre-point rule ($*$), respectively. From now on, we concentrate again on the one-dimensional case $d = 1$. This is sufficient for clarifying the basic concepts and ideas. The generalization to higher space-dimensions is straightforward and summarized in App. D.

Post-point discretization Similar to the nonrelativistic case, we postulate that the a momentum change $\mathrm{d}P(t)$ of the relativistic Brownian particle in the lab frame Σ (= rest frame of the bath) can be modeled by a Langevin equation of the from[5]

$$\mathrm{d}P(t) = -\alpha_\bullet(P)\,P\,\mathrm{d}t + [2D(P)]^{1/2} \bullet \mathrm{d}B(t), \tag{4.2a}$$

where $B(t)$ is a standard Wiener process with increment PDF (2.3d). If the background medium (heat bath) is *stationary* and *spatially homogeneous*, then both the friction coefficient α and the noise amplitude D should depend on the relativistic particle energy $E(P) = (M^2 + P^2)^{1/2}$ only, i.e.,

$$\alpha_\bullet(p) = \hat{\alpha}_\bullet(E)\ , \qquad\qquad D(p) = \hat{D}(E). \tag{4.2b}$$

In the remainder, we will always assume that Eqs. (4.2b) hold true. An additional constraint results from the requirement that Eq. (4.2a) should yield the correct stationary momentum distribution.

The FPE for the phase space density $f(t,x,p)$ of a relativistic Brownian particle, governed by Eq. (4.1) with $d = 1$ and Eq. (4.2a), reads

$$\frac{\partial f}{\partial t} + \frac{p}{E}\frac{\partial f}{\partial x} = \frac{\partial}{\partial p}\left[\alpha_\bullet(p)\,p\,f + D(p)\,\frac{\partial f}{\partial p}\right]. \tag{4.2c}$$

Analogous to Eq. (2.27), the corresponding stationary distribution is given by[6]

$$f_\infty(x,p) = \mathcal{N}\exp\left[-\int_{p_*}^{p}\mathrm{d}p'\,\frac{\alpha_\bullet(p')}{D(p')}\,p'\right]. \tag{4.2d}$$

[5] In principle, one could also consider other driving process (as, e.g., Levy or Poisson processes) and/or coefficient functions $\alpha(t,X,P)$ and $D(t,X,P)$.

[6] Here, we assume the presence of a spatial confinement or periodic boundary conditions.

The arbitrary boundary value p_* is absorbed by the normalization constant $\mathcal{N}$. Upon demanding that the marginal momentum distribution $\phi_\infty(p)$ be given by the relativistic Jüttner function $\phi_{\rm J}(p) \propto \exp[-\beta E(p)]$, we obtain the condition

$$\frac{\alpha_\bullet(p)}{D(p)}\, p \stackrel{!}{\equiv} -\frac{\mathrm{d}}{\mathrm{d}p} \log \phi_{\rm J}(p) = \frac{\mathrm{d}}{\mathrm{d}p} \beta E(p). \tag{4.2e}$$

Hence, by using $\mathrm{d}E(p)/\mathrm{d}p = p/E(p)$, we find

$$\frac{\alpha_\bullet(p)}{D(p)} \equiv \frac{\beta}{E(p)}. \tag{4.2f}$$

This is the *relativistic fluctuation-dissipation relation*, also referred to as the *relativistic Einstein relation*. Compared with the nonrelativistic Einstein relation (2.29), the mass has been replaced with energy on the rhs. of Eq. (4.2f).

Furthermore, it is straightforward to derive from Eq. (4.2a) the corresponding SDE for the relativistic energy $P^0 = E(P) = (M^2 + P^2)^{1/2}$. Applying the (backward) Ito formula (C.27) with $Y = P$, $Z = P^0$ and $G(p) = (M^2 + p^2)^{1/2}$, we obtain the following SDE for the relativistic energy $P^0 = E(P)$:

$$\begin{aligned} \mathrm{d}P^0(t) \;\; = \;\; & \left\{ -\hat{\alpha}_\bullet(P^0)\, P^0 \left[1 - \left(\frac{M}{P^0} \right)^2 \right] - \frac{\hat{D}(P^0)}{P^0} \left(\frac{M}{P^0} \right)^2 \right\} \mathrm{d}t + \\ & \left\{ 2\, \hat{D}(P^0) \left[1 - \left(\frac{M}{P^0} \right)^2 \right] \right\}^{1/2} \bullet\, \mathrm{d}B(t), \end{aligned} \tag{4.2g}$$

where $\alpha_\bullet(P) = \hat{\alpha}_\bullet(P^0)$ and $D(P) = \hat{D}(P^0)$.

Mid-point discretization The relativistic Langevin equation (4.2a) may be rewritten in terms of an equivalent Stratonovich-Fisk SDE, reading

$$\mathrm{d}P(t) = -\alpha_\circ(P)\, P\, \mathrm{d}t + [2D(P)]^{1/2} \circ \mathrm{d}B(t), \tag{4.3a}$$

where the friction coefficient functions $\alpha_\circ$ and $\alpha_\bullet$ are related by [cf. Eq. (C.43)][7]

$$\alpha_\circ(p)\, p = \alpha_\bullet(p)\, p - D'(p)/2 \tag{4.3b}$$

and $D'(p) := \mathrm{d}D(p)/\mathrm{d}p$. The relativistic Einstein relation (4.2f), reexpressed in terms of $\alpha_\circ(p)$, reads

$$\alpha_\circ(p) \equiv \frac{\beta D(p)}{E(p)} - \frac{1}{2}\, \frac{D'(p)}{p}, \tag{4.3c}$$

[7] See also the corresponding discussion by Hänggi [473], and Hänggi and Thomas (page 293 of Ref. [67]).

i.e., only if $\alpha_\circ(p)$ and $D(p)$ satisfy the criterion (4.3c), then the stationary momentum distribution is given by Jüttner's PDF. Moreover, if D and $\alpha_\bullet$ depend on the relativistic particle energy $P^0 = E$ only, i.e., if $D(p) = \hat{D}(E)$ and $\alpha_\bullet(p) = \hat{\alpha}_\bullet(E)$ hold true, then we may write

$$D'(p) = (p/E)\ \hat{D}'(E)$$

where $\hat{D}'(E) := \mathrm{d}\hat{D}(E)/\mathrm{d}E$. In this case, Eq. (4.3b) becomes equivalent to

$$\alpha_\circ(p) = \hat{\alpha}_\bullet(E) - \hat{D}'(E)/(2E) =: \hat{\alpha}_\circ(E), \tag{4.3d}$$

and the Einstein relation (4.3c) can be rewritten in the form

$$\hat{\alpha}_\circ(E) \equiv [2\beta\hat{D}(E) - \hat{D}'(E)]/(2E). \tag{4.3e}$$

The rules of ordinary differential calculus are preserved for Stratonovich-Fisk SDEs. Consequently, we find the following Stratonovich-Fisk SDE for the energy $P^0 = E(P)$:

$$\begin{aligned} \mathrm{d}P^0(t) \;&=\; -\hat{\alpha}_\circ(P^0)\,P^0\left[1-\left(\frac{M}{P^0}\right)^2\right]\mathrm{d}t + \\ &\qquad \left\{2\,\hat{D}(P^0)\left[1-\left(\frac{M}{P^0}\right)^2\right]\right\}^{1/2} \circ \mathrm{d}B(t). \end{aligned} \tag{4.3f}$$

Pre-point discretization The relativistic Langevin equations (4.2a) and (4.3a) can also be rewritten in terms of the equivalent Ito SDE

$$\mathrm{d}P(t) = -\alpha_*(P)\,P\,\mathrm{d}t + [2D(P)]^{1/2} * \mathrm{d}B(t), \tag{4.4a}$$

where the friction coefficients α_* and $\alpha_\bullet$ are related by [cf. Eq. (C.43)]

$$\alpha_*(p)\,p = \alpha_\bullet(p)\,p - D'(p). \tag{4.4b}$$

Compared with Eqs. (4.2a) and (4.3a), the Ito SDE (4.4) is most convenient for numerical simulations. For a homogeneous isotropic bath with $D(p) = \hat{D}(E)$ and $\alpha_\bullet(p) = \hat{\alpha}_\bullet(E)$, Eq. (4.4b) is equivalent to

$$\alpha_*(p) = \hat{\alpha}_\bullet(E) - \hat{D}'(E)/E =: \hat{\alpha}_*(E), \tag{4.4c}$$

and the relativistic Einstein relation becomes

$$\hat{\alpha}_*(E) \equiv [\beta\hat{D}(E) - \hat{D}'(E)]/E. \tag{4.4d}$$

Applying the Ito formula (C.10), we obtain the following Ito SDE for the energy P^0

$$\begin{aligned} \mathrm{d}P^0(t) &= \left\{ -\hat{\alpha}_*(P^0)\, P^0 \left[1 - \left(\frac{M}{P^0} \right)^2 \right] + \frac{\hat{D}(P^0)}{P^0} \left(\frac{M}{P^0} \right)^2 \right\} \mathrm{d}t + \\ &\qquad \left\{ 2\, \hat{D}(P^0) \left[1 - \left(\frac{M}{P^0} \right)^2 \right] \right\}^{1/2} * \mathrm{d}B(t). \end{aligned} \tag{4.4e}$$

Having outlined the general ideas underlying the axiomatic Langevin approach to relativistic Brownian motions in the lab frame, we next consider several example processes.

4.1.2 Examples

We discuss three specific 1D relativistic Langevin models whose stationary momentum distributions are given by the Jüttner function $\phi_{\mathrm{J}}(p) = \exp[-\beta E(p)]$. In this case, the relativistic Einstein relation (4.2f) implies that only one of the two functions $\alpha_\bullet(p)$ and $D(p)$ can be chosen arbitrarily.

Constant noise amplitude As a first example, we consider the so-called 'Relativistic Ornstein-Uhlenbeck process' (ROUP), proposed by Debbasch et al. [18,19] and also studied by Zygadło [334]. The ROUP is defined by the choice

$$\alpha_\bullet(p) = \alpha_c\, M/E(p), \tag{4.5a}$$

where $\alpha_c > 0$ is a constant friction parameter. From the relativistic Einstein relation (4.2f), one then finds

$$D(p) \equiv \alpha_\bullet(p)\, E(p)\, \beta^{-1} = \alpha_c M\, \beta^{-1} = \alpha_c M\, k_{\mathrm{B}} \mathcal{T} =: D_c, \tag{4.5b}$$

i.e., the ROUP corresponds to the limit case of constant noise amplitude. The associated Langevin equation reads

$$\mathrm{d}P(t) = -\alpha_c \frac{M}{P^0}\, P\, \mathrm{d}t + \left(\frac{2\alpha_c M}{\beta} \right)^{1/2} \bullet\, \mathrm{d}B(t) \tag{4.5c}$$

$$= -\alpha_c \frac{M}{P^0}\, P\, \mathrm{d}t + \left(\frac{2\alpha_c M}{\beta} \right)^{1/2} * \mathrm{d}B(t). \tag{4.5d}$$

The discretization rule is irrelevant here, because the noise amplitude $D_c = \alpha_c M/\beta$ does not depend on the momentum P for this particular case. However, the rules of stochastic calculus have to be specified, if one wishes to write down the SDE for the associated velocity

process $V(t) := P/P^0$. For example, adopting the post-point discretization, the Langevin equation for $V(t)$ reads

$$\mathrm{d}V(t) = -\alpha_c \left[(1-V^2)^{3/2} - \frac{3k_\mathrm{B}\mathcal{T}}{M}(1-V^2)^2\right] V\,\mathrm{d}t + \left[\frac{2\alpha_c k_\mathrm{B}\mathcal{T}}{M}\,(1-V^2)^3\right]^{1/2} \bullet\, \mathrm{d}B(t). \tag{4.5e}$$

For comparison, the corresponding Ito SDE is given by

$$\mathrm{d}V(t) = -\alpha_c \left[(1-V^2)^{3/2} + \frac{3k_\mathrm{B}\mathcal{T}}{M}(1-V^2)^2\right] V\,\mathrm{d}t + \left[\frac{2\alpha_c k_\mathrm{B}\mathcal{T}}{M}\,(1-V^2)^3\right]^{1/2} * \mathrm{d}B(t). \tag{4.5f}$$

Constant friction coefficient in the backward-Ito SDE An alternative relativistic Brownian motion (RBM) model, considered by us in Ref. [10,11], corresponds to the special case of a constant friction function $\alpha_\bullet(p) \equiv \alpha_\dagger$ in the backward-Ito SDE (4.2a). In this case, the relativistic Einstein relation (4.2f) yields the momentum dependent noise amplitude

$$D(p) = \alpha_\dagger E(p)\,\beta^{-1}. \tag{4.6a}$$

Adopting the post-point discretization scheme, the relativistic Langevin equation of this model reads

$$\mathrm{d}P(t) = -\alpha_\dagger\, P\,\mathrm{d}t + \left(\frac{2\alpha_\dagger P^0}{\beta}\right)^{1/2} \bullet\, \mathrm{d}B(t). \tag{4.6b}$$

The corresponding SDE for the velocity process $V(t) = P/P^0$ is given by

$$\mathrm{d}V(t) = -\alpha_\dagger \left[(1-V^2) - \frac{3k_\mathrm{B}\mathcal{T}}{M}(1-V^2)^{3/2}\right] V\,\mathrm{d}t + \left[\frac{2\alpha_c k_\mathrm{B}\mathcal{T}}{M}\,(1-V^2)^{5/2}\right]^{1/2} \bullet\, \mathrm{d}B(t). \tag{4.6c}$$

Recently, various properties of the RBM process (4.6b) have been analyzed by Fa [331], Lindner [24], Fingerle[8] [470], and Angst and Franchi [23].

With regard to numerical simulations,[9] the Ito form of Eqs. (4.6b) is often more convenient. By making use of Eq. (4.4c), the equivalent Ito SDEs are obtained as

$$\mathrm{d}P(t) = -\alpha_\dagger \left(\frac{\beta P^0 - 1}{\beta P^0}\right) P\,\mathrm{d}t + \left(\frac{2\alpha_\dagger P^0}{\beta}\right)^{1/2} * \mathrm{d}B(t), \tag{4.6d}$$

[8] Fingerle [470] discusses a fluctuation theorem for this process; see also Cleuren et al. [218].

[9] Cf. App. C.

and

$$\mathrm{d}V(t) = -\alpha_\dagger \left[(1-V^2) + \frac{2k_\mathrm{B}\mathcal{T}}{M}(1-V^2)^{3/2}\right] V\,\mathrm{d}t + \left[\frac{2\alpha_c k_\mathrm{B}\mathcal{T}}{M}\,(1-V^2)^{5/2}\right]^{1/2} * \mathrm{d}B(t). \tag{4.6e}$$

Constant friction coefficient in the Ito-SDE The RBM process defined by Eq. (4.6b) is characterized by a constant friction coefficient $\alpha_\dagger$, when adopting the post-point discretization rule ($\bullet$). Another model, referred to as RBM(I) hereafter, is obtained by considering a constant friction coefficient α_* in the Ito-Langevin equation

$$\mathrm{d}P(t) = -\alpha_*\, P\,\mathrm{d}t + \left[\frac{2\alpha_*}{\beta^2}(1+\beta P^0)\right]^{1/2} * \mathrm{d}B(t), \tag{4.7a}$$

where the noise amplitude is chosen such that the Einstein relation (4.4d) is satisfied. For completeness, the Ito-SDE of the associated velocity process $V(t) := P/P^0$ is given by

$$\mathrm{d}V(t) = -\alpha_* \left[(1-V^2) + 3\frac{k_\mathrm{B}\mathcal{T}}{M}(1-V^2)^{3/2} + 3\left(\frac{k_\mathrm{B}\mathcal{T}}{M}\right)^2 (1-V^2)^2\right] V\,\mathrm{d}t + \left\{2\alpha_*\frac{k_\mathrm{B}\mathcal{T}}{M}\,\left[(1-V^2)^{5/2} + \frac{k_\mathrm{B}\mathcal{T}}{M}(1-V^2)^3\right]\right\}^{1/2} * \mathrm{d}B(t). \tag{4.7b}$$

The three model processes (4.5d), (4.6b) and (4.7a) give rise to the same stationary momentum PDF $\phi_\mathrm{J}(p)$, but their respective relaxation behavior differs strongly. This is illustrated in Fig. 4.1, which depicts the time evolution of the spatial mean square displacement divided by time,

$$\mathcal{D}_t := \left\langle [X(t) - X(0)]^2 \right\rangle / (2t), \tag{4.8}$$

for all three models at same temperature $\mathcal{T}$. The curves in Fig. 4.1 were calculated numerically from Eqs. (4.5d), (4.6d) and (4.7a), respectively, using an algorithm similar to those described in [23, 24], see also App. C.5.

4.1.3 Asymptotic mean square displacement

A primary objective within any Brownian theory is to determine the asymptotic diffusion constant $\mathcal{D}_\infty$, corresponding to the plateau values in Fig. 4.1. For a 1D diffusion process $X(t)$ with velocity $V(t)$, the asymptotic diffusion constant $\mathcal{D}_\infty$ is defined by

$$\mathcal{D}_\infty = \lim_{t\to\infty} \left\langle [X(t) - X(0)]^2 \right\rangle / (2t), \tag{4.9a}$$

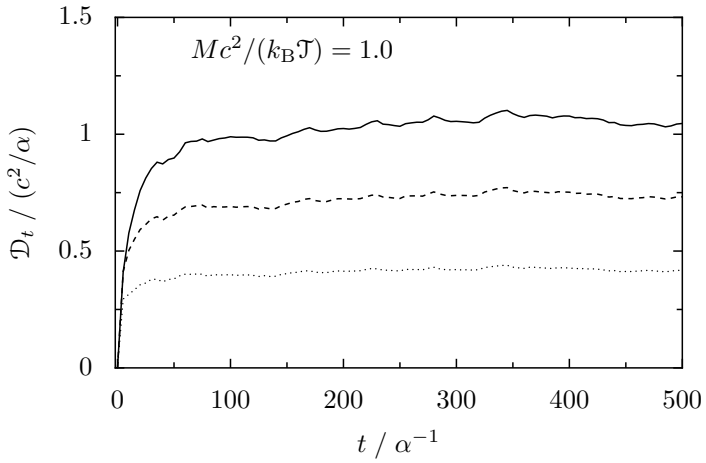


Figure 4.1: Time evolution of the spatial mean square displacement $\mathcal{D}_t := \left\langle [X(t) - X(0)]^2 \right\rangle / (2t)$ for the ROUP [18] model (solid line) from Eq. (4.5d), the RBM [10] model (dotted) from Eq. (4.6b) and the RBM(I) model (dashed) from Eq. (4.7a) at same temperature $k_B\mathcal{T}/(Mc^2) = 1$. The plots are based on a simulation with $N = 1000$ trajectories, initial conditions $X(0) = 0$, $P(0) = 0$ for each trajectory, and discretization time step $\Delta t = 10^{-4}\ \alpha_{c/\dagger}^{-1}$.

where the spatial displacement is given by

$$X(t) - X(0) = \int_0^t \mathrm{d}s\ V(s). \tag{4.9b}$$

The asymptotic diffusion constant $\mathcal{D}_\infty$ may be expressed in terms of the velocity correlation function $\langle V(t)V(s)\rangle$ by virtue of

$$\begin{aligned} \mathcal{D}_\infty &= \lim_{t\to\infty} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \left\langle [X(t) - X(0)]^2 \right\rangle \\ &= \lim_{t\to\infty} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \mathrm{d}s \int_0^t \mathrm{d}s'\ \langle V(s)V(s')\rangle \\ &= \lim_{t\to\infty} \int_0^t \mathrm{d}s\ \langle V(t)V(s)\rangle\,. \end{aligned} \tag{4.10}$$

Assuming that the velocity process $V(t)$ is (approximately) stationary, which means that $\langle V(t)V(s)\rangle = \langle V(t-s)V(0)\rangle$ holds (at least in good approximation), and substituting $u = t - s$, we recover Kubo's formula

$$\mathcal{D}_\infty = \lim_{t\to\infty} \int_0^t \mathrm{d}u\ \langle V(u)V(0)\rangle\,. \tag{4.11}$$

As recently discussed by Lindner [24], for a 1D Langevin equation of the form

$$\mathrm{d}V(t) = -a_\bullet(V)\, V\, \mathrm{d}t + [2\, b(V)]^{1/2} \bullet \mathrm{d}B(t) \tag{4.12}$$

with symmetric coefficient functions, $a_\bullet(v) = a_\bullet(-v)$ and $b(v) = b(-v)$, the Kubo formula (4.11) gives rise to the following integral representation for the asymptotic diffusion constant:[10]

$$\mathcal{D}_\infty = \frac{\int_0^{v_+} \mathrm{d}y\; e^{U(y)} \left[\int_y^{v_+} \mathrm{d}x\; e^{-U(x)}\, x/b(x)\right]^2}{\int_0^{v_+} \mathrm{d}z\; e^{-U(z)}/b(z)}. \tag{4.13}$$

Here, $v_+ \in [0,\infty]$ represents the upper bound for the velocity range, and

$$U(v) := \int_0^v \mathrm{d}w\; \mu_*(w)/b(w) \tag{4.14a}$$

is an effective velocity potential with Ito drift

$$\mu_*(v) \;:=\; a_*(v)\, v = a_\bullet(v)\, v - b'(v). \tag{4.14b}$$

In general, the formula (4.13) has to be integrated numerically, but for the first two models from Section 4.1.2 the integrals may also be evaluated analytically.

For example, upon comparing with Eq. (4.6e), we see that the RBM model from Eq. (4.6b) is described by

$$\mu_*(v) \;=\; \alpha_\dagger \left[(1-v^2) + \frac{2k_\mathrm{B}\mathcal{T}}{M}\,(1-v^2)^{3/2}\right] v, \tag{4.15a}$$

$$b(v) \;=\; \alpha_\dagger\, \frac{k_\mathrm{B}\mathcal{T}}{M}\,(1-v^2)^{5/2}, \tag{4.15b}$$

with an upper velocity bound $v_+ = c = 1$. In this case, the rhs. of Eq. (4.13) can be calculated analytically by making use of the identity

$$(-1)^\nu \frac{\mathrm{d}^\nu}{\mathrm{d}\chi^\nu} K_0(\chi) \;=\; \int_0^1 \mathrm{d}v\; \exp\!\left(-\frac{\chi}{\sqrt{1-v^2}}\right) (1-v^2)^{-(\nu+2)/2}, \tag{4.16}$$

where, for $\nu = 0, 1, 2\ldots$, $K_\nu(z)$ denotes the modified Bessel function of the second kind [466]. One then finds that[11]

$$\mathcal{D}_\infty^{\mathrm{RBM}} = (\alpha_\dagger \beta M)^{-1}\, \frac{K_0(\beta M)}{K_1(\beta M)}. \tag{4.17}$$

At low temperatures $\beta := (k_\mathrm{B}\mathcal{T})^{-1} \to \infty$, Eq. (4.17) reduces to the well-known classical result $\mathcal{D}_\infty^{\mathrm{class}} = k_\mathrm{B}\mathcal{T}/(M\alpha_\dagger)$, cf. Eq. (2.20). In the opposite limit of very high temperatures, i.e., for $\beta M \ll 1$, one finds a logarithmic dependence [23]

$$\mathcal{D}_\infty^{\mathrm{RBM}} = (\alpha_\dagger M)^{-1} \left\{-\gamma_\epsilon + \ln\!\big[\frac{2}{\beta M}\big] + \mathcal{O}[(\beta M)^2]\right\}, \tag{4.18}$$

[10] A d-dimensional generalization of Eq. (4.13) was recently derived by Angst and Franchi [23].

[11] Equation (4.17) is an equivalent, more compact representation of Lindner's result Eq. (10) in [24].

where $\gamma_\epsilon \simeq 0.577216$ is the Euler constant. However, it should be kept in mind that, due to the increasing importance of particle annihilation/creation at high energies, classical non-quantum theories become invalid in the high temperature limit $\beta M \ll 1$, and, therefore, the asymptotic expansion (4.18) appears to be of limited practical use.

For comparison, the ROUP (4.5d) corresponds to [cf. Eq. (4.5f)]

$$\mu_*(v) = \alpha_c \left[(1-v^2)^{3/2} + \frac{3k_{\rm B}\mathcal{T}}{M}(1-v^2)^2\right] v, \tag{4.19a}$$

$$b(v) = \alpha_c \frac{k_{\rm B}\mathcal{T}}{M}(1-v^2)^3. \tag{4.19b}$$

Remarkably, in this case the general integral formula (4.13) and the identity (4.16) yield the 'classical' result

$$\mathcal{D}_\infty^{\rm ROUP} = k_{\rm B}\mathcal{T}/(M\alpha_c) = (\alpha_c \beta M)^{-1} \tag{4.20}$$

for all parameter values $(\alpha_c, \mathcal{T}, M)$. Moreover, we observe that $\mathcal{D}_\infty^{\rm RBM} \leq \mathcal{D}_\infty^{\rm ROUP}$ holds true for same values of the friction coefficients $\alpha_c = \alpha_\dagger$. Intuitively, this can be explained by the fact that, for the ROUP, the absolute value of the friction force is bounded by $\alpha_c M$, cf. Eq. (4.5d), whereas the friction force is unbounded for the RBM model (4.6b), thereby suppressing spatial diffusion more strongly in the latter case.

As the last example, we consider the RBM(I) model, defined in Eq. (4.7a) and described by [cf. Eq. (4.7b)]

$$\mu_*(v) = \alpha_* \left[(1-v^2) + 3\frac{k_{\rm B}\mathcal{T}}{M}(1-v^2)^{3/2} + 3\left(\frac{k_{\rm B}\mathcal{T}}{M}\right)^2 (1-v^2)^2\right] v, \tag{4.21a}$$

$$b(v) = \alpha_* \frac{k_{\rm B}\mathcal{T}}{M}\left[(1-v^2)^{5/2} + \frac{k_{\rm B}\mathcal{T}}{M}(1-v^2)^3\right]. \tag{4.21b}$$

From these equations the velocity potential is obtained as

$$U(v) = \ln\left[\frac{\beta M + 1}{\beta M(1-v^2) + (1-v^2)^{3/2}}\right] - \beta M[1 - (1-v^2)^{-1/2}],$$

yielding for the asymptotic diffusion constant:

$$\mathcal{D}_\infty^{\rm RBM(I)} = [\alpha_* K_1(\beta M)]^{-1} \int_0^1 {\rm d}v \, \frac{e^{-\beta M(1-v^2)^{-1/2}}}{\beta M(1-v^2) + (1-v^2)^{3/2}}. \tag{4.22}$$

The remaining integral can be evaluated numerically. As illustrated in Fig. 4.2, the theoretical predictions from Eqs. (4.17), (4.20) and (4.22) are in good agreement with the numerically obtained estimates of the asymptotic diffusion constant.

The ROUP and the two RBM models considered in this part represent special limit cases of the general Langevin equation (4.2a) with arbitrarily chosen friction coefficient functions. Nonetheless, they yield useful insights: As evident from Fig. 4.2, at moderate-to-high temperatures the diffusion constant can vary significantly for different friction models. For realistic systems, the exact functional shape (i.e., energy dependence) of the friction function α is determined by the microscopic interactions. This result implies that simultaneous measurements of the temperature and the diffusion constants can reveal information about the underlying microscopic forces. Below, in Section 4.3, we will outline a general procedure for deducing more realistic friction coefficients α from microscopic models.

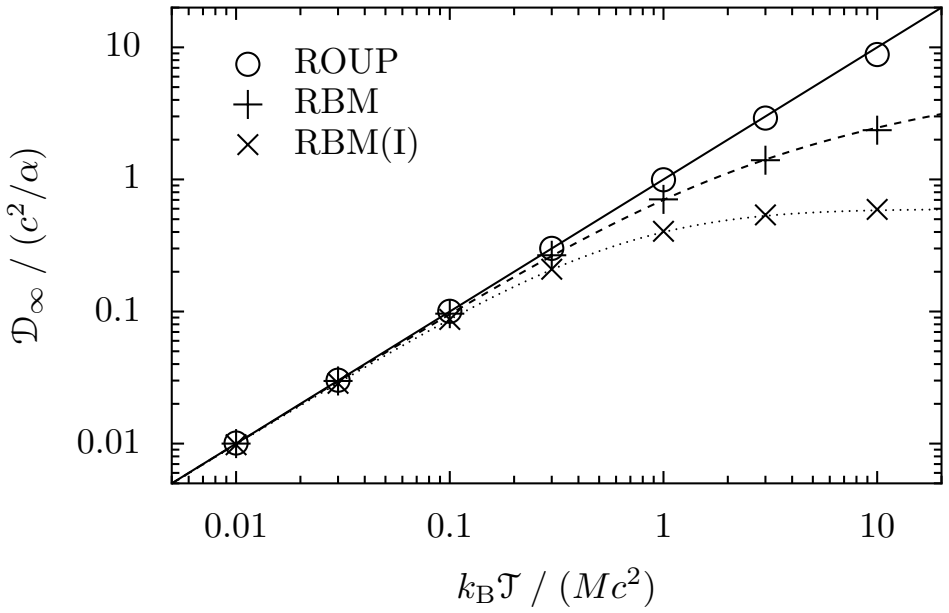


Figure 4.2: Temperature dependence of the asymptotic diffusion constant $\mathcal{D}_\infty$ for the ROUP [18] from Eq. (4.5d), the RBM model [10] from Eq. (4.6b), and the RBM(I) model from Eq. (4.7a). Symbols '$\circ$', '$+$' and '$\times$' represent the results of computer simulations ($N = 100$ trajectories, initial conditions $X(0) = 0$, $P(0) = 0$ for each trajectory, discretization time step $\Delta t = 10^{-4}$ in units of $\alpha^{-1}_{c/\dagger/*}$), obtained by averaging over the numerically determined values $\{\mathcal{D}_{100}, \mathcal{D}_{110}, \mathcal{D}_{120}, \ldots, \mathcal{D}_{500}\}$. Solid, dashed and dotted lines correspond to the theoretical predictions from Eqs. (4.20), (4.17) and (4.22), respectively.

4.2 Moving observer

Thus far we have focussed on Langevin equations that describe the random motion of a relativistic Brownian particle in the lab frame, defined as the rest frame of the heat bath. In this part we would like to address the following question: Assuming that a Langevin equation of the type (4.2a) holds in the lab frame – how does the corresponding process look from the viewpoint of a moving observer?

To clarify this for the case of one spatial dimension (1D) we start from the Langevin equations in the bath frame Σ,

$$\begin{aligned} \mathrm{d}X(t) &= (P/P^0)\,\mathrm{d}t, && \text{(4.23a)} \\ \mathrm{d}P(t) &= -\alpha(P)\,P\,\mathrm{d}t + [2D(P)]^{1/2} \bullet \mathrm{d}B(t), && \text{(4.23b)} \end{aligned}$$

and demand that α and D satisfy the relativistic Einstein relation $\beta D(p) = \alpha(p)E(p)$ with $E(P) = (M^2 + P^2)^{1/2} = P^0$. Then, upon multiplying by $p^0 = (M^2 + p^2)^{1/2}$, the FPE (4.2c) for the associated PDF $f(t, x, p)$ can be written in the form

$$\left(p^0 \frac{\partial}{\partial t} + p \frac{\partial}{\partial x}\right) f = p^0 \frac{\partial}{\partial p}\left[\alpha(p)\, p\, f + D(p) \frac{\partial}{\partial p} f\right]. \tag{4.24}$$

As before, we will assume deterministic initial conditions $X(0) = x_0$, $P(0) = p_0$ for the stochastic process (4.23), corresponding to a localized initial distribution $f(0, x, p) = \delta(x - x_0)\,\delta(p - p_0)$ in Σ.

In order to characterize the process (4.23) from the viewpoint of a moving inertial frame Σ', one can proceed as follows: According to van Kampen [138], the one-particle phase space PDF f transforms as a Lorentz scalar, i.e., we have

$$f'(t', x', p') = f(t(t', x'), x(t', x'), p(p')), \tag{4.25a}$$

and, conversely,

$$f(t, x, p) = f'(t'(t, x), x'(t, x), p'(p)), \tag{4.25b}$$

where (t', x', p') and (t, x, p) are related by the Lorentz transformations

$$\begin{aligned} \begin{pmatrix} t' \\ x' \end{pmatrix} &= \gamma(w) \begin{pmatrix} t - wx \\ -wt + x \end{pmatrix}, \qquad \begin{pmatrix} p'^0 \\ p' \end{pmatrix} = \gamma(w) \begin{pmatrix} p^0 - wp \\ -wp^0 + p \end{pmatrix}, && \text{(4.26a)} \\ \begin{pmatrix} t \\ x \end{pmatrix} &= \gamma(w) \begin{pmatrix} t' + wx' \\ wt' + x' \end{pmatrix}, \qquad \begin{pmatrix} p^0 \\ p \end{pmatrix} = \gamma(w) \begin{pmatrix} p'^0 + wp' \\ wp'^0 + p' \end{pmatrix}, && \text{(4.26b)} \end{aligned}$$

with w denoting the velocity of Σ' relative to Σ, $\gamma(w) = (1 - w^2)^{-1/2}$ and $p'^0 = (M^2 + p'^2)^{1/2}$. Thus, in order to find f', one merely needs to solve the Fokker-Planck equation (4.24) in the lab frame Σ and, subsequently, insert the solution into Eq. (4.25a).

4.2.1 Fokker-Planck and Langevin equations

On the other hand, it is also interesting to derive an explicit evolution equation for $f'(t', x', p')$ by starting from the lab frame FPE (4.24). To this end, we note that the lhs. of Eq. (4.24) can be transformed as[12]

$$\left(p^0 \frac{\partial}{\partial t} + p \frac{\partial}{\partial x}\right) f = p^\alpha \partial_\alpha f = p'^\beta \, \partial'_\beta \, f' = \left(p'^0 \frac{\partial}{\partial t'} + p' \frac{\partial}{\partial x'}\right) f', \tag{4.27}$$

where $(\partial_\alpha) := (\partial/\partial t, \partial/\partial x)$ and $(p^\alpha) := (p^0, p)$. The rhs. of Eq. (4.24) may be rewritten in terms of the primed quantities as

$$p^0 \frac{\partial}{\partial p}\left[\alpha(p)\, p\, f + D(p)\, \frac{\partial}{\partial p} f\right] \;=\; p^0 \frac{\partial p'}{\partial p} \frac{\partial}{\partial p'}\left[\alpha(p(p'))\, p(p')\, f' + D(p(p'))\, \frac{\partial p'}{\partial p} \frac{\partial}{\partial p'} f'\right], \tag{4.28a}$$

where, by virtue of Eqs. (4.26), we have in the 1D case

$$p(p') = \gamma(w)\, (w p'^0 + p'), \qquad\qquad \frac{\partial p'}{\partial p} = \frac{p'^0}{\gamma(w)\,(p'^0 + w p')} = \frac{p'^0}{p^0}. \tag{4.28b}$$

Defining

$$\alpha'(p') := \alpha(p(p')), \qquad\qquad D'(p') := D(p(p')), \tag{4.28c}$$

and combining Eqs. (4.27) and (4.28), we thus find the following FPE for the moving observer

$$p'^\beta \, \partial'_\beta \, f' = p'^0 \frac{\partial}{\partial p'}\left[\alpha'(p')\, \gamma(w)\, (w p'^0 + p')\, f' + \frac{D'(p')\, p'^0}{\gamma(w)\,(p'^0 + w p')} \frac{\partial}{\partial p'} f'\right]. \tag{4.29}$$

Finally, upon dividing by p'^0, this may be cast in the more familiar Fokker-Planck form

$$\left(\frac{\partial}{\partial t'} + \frac{p'}{p'^0}\, \frac{\partial}{\partial x'}\right) f' = \frac{\partial}{\partial p'}\left[\alpha'(p')\, \gamma(w)\, (w p'^0 + p')\, f' + \frac{D'(p')\, p'^0}{\gamma(w)\,(p'^0 + w p')} \frac{\partial}{\partial p'} f'\right]. \tag{4.30}$$

The post-point SDE corresponding to this FPE reads

$$\mathrm{d}X'(t') \;=\; (P'/P'^0)\, \mathrm{d}t', \tag{4.31a}$$

$$\mathrm{d}P'(t') \;=\; A(P'; w)\, \mathrm{d}t' + C(P'; w) \bullet \mathrm{d}B'(t'), \tag{4.31b}$$

where $B'(t')$ now is a standard Wiener process with time parameter t', and

$$A(p'; w) := -\alpha'(p')\, \gamma(w)\, (w p'^0 + p'), \qquad C(p'; w) := \left[2\, \frac{D'(p')\, p'^0}{\gamma(w)\,(p'^0 + w p')}\right]^{1/2}. \tag{4.31c}$$

Equations (4.31) describe how the stochastic process (4.23) would look like from the standpoint of a moving observer who sees the lab frame flying past at velocity $-w$. In particular, for $w = 0$ Eqs. (4.31) coincide with the lab frame Langevin equation (4.23).[13]

[12] Here and below, we use the sum convention $a^\alpha b_\alpha := \sum_\alpha a^\alpha b_\alpha$.

[13] We note that, in general, a t-simultaneously specified initial condition $f(0, \boldsymbol{x}, \boldsymbol{p})$ in the lab frame Σ corresponds to a non-simultaneous initial condition in the moving frame Σ' – unless one considers a strictly

4.2.2 Covariant formulation

The FPE (4.30) describes the process (4.23) in the moving frame, but is not yet written a manifestly covariant form. In order to achieve this, we return to Eq. (4.29). Dropping the primes and writing $f(t,x,p;w)$, $\alpha(p^0,p;w)$ and $D(p^0,p;w)$ instead, Eq. (4.29) becomes

$$p^\beta\,\partial_\beta\,f = p^0\frac{\partial}{\partial p}\left[\alpha(p^0,p;w)\,\gamma(w)\,(wp^0+p)\,f + \frac{D(p^0,p;w)}{\gamma(w)\,(p^0+wp)}\,p^0\frac{\partial}{\partial p}f\right], \tag{4.32}$$

with $(p^\alpha) = (p^0,p)$ denoting the contravariant momentum four-vector of the Brownian particle and $p^0 = (M^2+p^2)^{1/2}$. If an observer moves at velocity w relative to the lab frame, defined as mean rest frame of the bath, then from her point of view the heat bath moves at velocity $-w$. Introducing the contravariant mean velocity four-vector of the heat bath by $(U^\alpha) := \gamma(w)(1,-w)$, we may simplify

$$\gamma(w)\,(p^0+wp) = -U^\alpha p_\alpha. \tag{4.33}$$

Since f,α and D transform as Lorentz scalars and because of $p^0\partial/\partial p = p'^0\partial/\partial p'$ in the 1D case, we can already see that the rhs. of Eq. (4.32) is indeed Lorentz invariant. However, to obtain a manifestly covariant form of Eq. (4.32), we consider the momentum derivative of some Lorentz scalar function $g(p) = G(p^0(p),p)$ and note that

$$\begin{aligned}
p^0\frac{\partial}{\partial p}g(p) &= p^0\left\{\frac{\partial}{\partial p}G(p^0,p) + \left[\frac{\partial}{\partial p^0}G(p^0,p)\right]\frac{\partial p^0}{\partial p}\right\}\\
&= p^0\left\{\frac{\partial}{\partial p}G(p^0,p) + \left[\frac{\partial}{\partial p^0}G(p^0,p)\right]\frac{p}{p^0}\right\}\\
&= p^0\frac{\partial}{\partial p}G(p^0,p) + p\frac{\partial}{\partial p^0}G(p^0,p)\\
&= -p_0\frac{\partial}{\partial p}G(p^0,p) + p\frac{\partial}{\partial p^0}G(p^0,p)\\
&= -\varepsilon^{\alpha\beta}p_\alpha\frac{\partial}{\partial p^\beta}G(p^0,p).
\end{aligned} \tag{4.34}$$

Here, $(p_\alpha) = (p_0,p) = (-p^0,p)$ is the covariant momentum four-vector and the total antisymmetric Levi-Cevita tensor $\varepsilon^{\alpha\beta}$ is defined by[14]

$$\varepsilon^{01} = -\varepsilon^{10} = 1\,, \qquad\qquad \varepsilon^{00} = \varepsilon^{11} = 0.$$

Thus, by means of Eq. (4.34), Eq. (4.32) can be written in the Lorentz invariant form

$$p^\beta\,\partial_\beta\,f = \varepsilon^{\alpha\beta}p_\alpha\frac{\partial}{\partial p^\beta}\left[\alpha(p^\nu;w)\,U^\eta p_\eta\,f + \frac{D(p^\mu;w)}{-U^\kappa p_\kappa}\varepsilon^{\gamma\delta}p_\gamma\frac{\partial}{\partial p^\delta}f\right]. \tag{4.35}$$

localized initial condition of the form $f(0,\boldsymbol{x},\boldsymbol{p}) = \delta(\boldsymbol{x}-\boldsymbol{x}_0)\delta(\boldsymbol{p}-\boldsymbol{p}_0)$.

[14]For a Lorentz boost of the form (3.6), one finds that $\varepsilon'^{\alpha\beta} = \Lambda^\alpha{}_\gamma\Lambda^\beta{}_\delta\varepsilon^{\gamma\delta} = \varepsilon^{\gamma\delta}$, i.e., the Levi-Cevita tensor is numerically invariant under Lorentz transformations with determinant +1. For a general discussion of the properties of Levi-Cevita tensors we refer to Section 5.5 of Sexl and Urbantke [8].

4.3 Relativistic binary collision model

The two preceding sections have focussed on general aspects of relativistic Langevin and Fokker-Planck equations. Similar to the nonrelativistic case, relativistic SDEs present a useful tool for analytical and numerical studies of relaxation processes in relativistic systems. Stochastic models of this type provide a simplified picture of the underlying microscopic dynamics. In order for the Langevin approach to be successful, one must know in advance which friction coefficient function $\alpha(P)$ and noise amplitude $D(P)$ are appropriate for the system under consideration. In the remainder of this section, we will discuss a systematic procedure for obtaining friction coefficients and noise amplitudes from a simple microscopic interaction model [13]. The latter can be viewed as the direct relativistic generalization of the elastic binary collision model from Section 2.2.2. More precisely, we consider a 1D system consisting of a heavy Brownian particle (mass M) which is embedded into a heat bath of smaller particles (mass $m \ll M$, total number $N \gg 1$). Our model assumes that the stochastic motion of a Brownian particle arises due to frequent elastic interactions with the surrounding heat bath particles. Similar to Section 2.2.2, we are interested in finding the 'best' approximation of the 'exact' dynamics within the class of SDEs defined by Eq. (4.2a).

Relativistic collision kinematics To begin with, we consider a single collision of the Brownian particle (momentum P, energy E) with a heat bath particle (momentum p, energy ϵ). The relativistic energy, momentum and velocity of the two particles before the collision are given by

$$P = MV\,\gamma(V), \qquad E(P) = \left(M^2 + P^2\right)^{1/2}, \tag{4.36a}$$

$$p = mv\,\gamma(v), \qquad \epsilon(p) = \left(m^2 + p^2\right)^{1/2}. \tag{4.36b}$$

where $\gamma(v) \equiv (1-v^2)^{-1/2}$. Considering elastic interactions, the collision kinematics is governed by the relativistic mass-energy-momentum conservation laws

$$\hat{M} = M, \qquad \hat{m} = m, \qquad E + \epsilon = \hat{E} + \hat{\epsilon}, \qquad P + p = \hat{P} + \hat{p}, \tag{4.37}$$

where hat-symbols refer to the state after the collision. Inserting Eqs. (4.36) into the conservation laws (4.37), and solving for the momentum of Brownian particle after the collision, $\hat{P}$, we obtain [13]

$$\hat{P} \;=\; \gamma(u)^2\,[2u\,E - (1+u^2)\,P], \tag{4.38a}$$

where the collision-invariant center-of-mass velocity u is given by

$$u(p,P) = \frac{P+p}{E+\epsilon}. \tag{4.38b}$$

Accordingly, the momentum change

$$\Delta P_r := \hat{P} - P$$

of the Brownian particle in a single collision with the heat bath particle 'r' is given by

$$\Delta P_r = -2\gamma(u_r)^2 \, \frac{\epsilon_r}{E+\epsilon_r} \, P + 2\gamma(u_r)^2 \, \frac{E}{E+\epsilon_r} \, p_r, \tag{4.39}$$

where $u_r := u(p_r, P)$ and $\epsilon_r := \epsilon(p_r)$. In the non-relativistic limit case, where $u_r^2 \ll 1$, $E \simeq M$ and $\epsilon_r \simeq m$, Eq. (4.39) reduces to Eq. (2.43).

Furthermore, by making the same assumptions as in Section 2.2.2, we find that the momentum change $\delta P(t) := P(t+\delta t) - P(t)$ of the Brownian particle during a small-but-sufficiently-long time interval $[t, t+\delta t]$ can be approximated by

$$\begin{aligned} \delta P(t) \;&\approx\; \sum_{r=1}^{N} \Delta P_r \; I_r(t, \delta t) \\ &\approx\; -2 \sum_{r=1}^{N} \gamma(u_r)^2 \, \frac{\epsilon_r}{E+\epsilon_r} \, P(t) \; I_r(t, \delta t) + 2 \sum_{r=1}^{N} \gamma(u_r)^2 \, \frac{E}{E+\epsilon_r} \, p_r \; I_r(t, \delta t). \end{aligned} \tag{4.40a}$$

Formally, the collision indicator $I_r(t, \tau)$ is again given by [cf. Eq. (2.48a)]

$$I_r(t, \delta t) \;\approx\; \frac{\delta t}{2} \, |v_r - V| \, \delta(x_r - X), \tag{4.40b}$$

but now we have to use the relativistic velocities

$$V = P/(M^2 + P^2)^{1/2} \,, \qquad\qquad v_r = p_r/(M^2 + p_r^2)^{1/2}. \tag{4.40c}$$

Equation (4.40a) is the relativistic counterpart of Eq. (2.48b). Heuristically, the first term on the rhs. of Eq. (4.40a) can again be interpreted as 'friction', while the second contribution may be viewed as 'noise'.

Bath distribution and drift Similar to the nonrelativistic case, Eqs. (4.40) can be used to calculate the statistical moments $\langle [\delta P(t)]^j \rangle_{\rm b}$ of the momentum increments – provided one specifies the phase space distribution of the heat bath particles. We will assume here that the heat bath is in a thermal equilibrium state, so that the one-particle phase space PDF is given by a spatially homogeneous Jüttner function

$$f_{\rm b}^1(x_r, p_r) = (\mathcal{Z}_{\rm J} L)^{-1} \exp\left[-\frac{(m^2 + p_r^2)^{1/2}}{k_{\rm B} \mathcal{T}} \right], \tag{4.41a}$$

where L is the 1D volume, $\mathcal{T} = (\beta k_{\rm B})^{-1}$ the temperature, and

$$\mathcal{Z}_{\rm J} = 2m \, K_1(\beta m), \tag{4.41b}$$

with $K_1(z)$ denoting the modified Bessel function. With regard to our subsequent discussion, we are interested in calculating the mean drift force g, defined by[15]

$$g(P) := \left\langle \frac{\delta P(t)}{\delta t} \right\rangle_{\rm b}. \tag{4.42}$$

Inserting $\delta P(t)$ from Eq. (4.40a), we find

$$g(P) = -n_{\rm b} \left\langle 2\gamma(u_r)^2 \, \frac{\epsilon_r}{E+\epsilon_r} \, \frac{LI_r}{\delta t} \right\rangle_{\rm b} P + n_{\rm b} \left\langle 2\gamma(u_r)^2 \, \frac{E}{E+\epsilon_r} \, p_r \, \frac{LI_r}{\delta t} \right\rangle_{\rm b}, \tag{4.43}$$

where $n_{\rm b} = N/L$ is the number density of the bath particles. In order to determine $g(P)$, we note that for some arbitrary function $G(p,P)$, we have

$$\begin{aligned} \left\langle G(p_r,P) \, \frac{LI_r(t,\delta t)}{\delta t} \right\rangle_{\rm b} &= (2\mathcal{Z}_{\rm J})^{-1} \int_{-\infty}^{\infty} {\rm d}p \; G(p_r,P) \; \exp\left[-\frac{(m^2+p_r^2)^{1/2}}{k_{\rm B}\mathcal{T}} \right] \times \\ &\qquad \left| \frac{p_r}{(m^2+p_r^2)^{1/2}} - \frac{P}{(M^2+P^2)^{1/2}} \right|. \end{aligned} \tag{4.44}$$

The first term on the rhs. of Eq. (4.43) involves the function

$$G_1(p,P) := 2\gamma(u_r)^2 \, \frac{\epsilon(p)}{E(P)+\epsilon(p)}, \tag{4.45a}$$

and the second term

$$G_2(p,P) := 2\gamma(u_r)^2 \, \frac{E(P)}{E(P)+\epsilon(p)} \, p. \tag{4.45b}$$

Unfortunately, it is very difficult or perhaps even impossible to analytically evaluate the integral (4.44) for the functions $G_{1/2}$. Figure 4.3 depicts the mean drift force $g(P)$, obtained by numerically integrating the formula (4.44) for different values of P.

Langevin approximation We conclude this section by discussing how one could, in principle, approximate Eqs. (4.40) by a nonlinear SDE of the form (4.2a), i.e., by

$${\rm d}P(t) = -\alpha(P) \, P \, {\rm d}t + [2D(P)]^{1/2} \bullet {\rm d}B(t). \tag{4.46}$$

The considerations from Chapter 3 imply that the stationary momentum distribution of the Brownian particle in the binary collision model is given by the Jüttner function

$$\phi_{\rm J}(p) = \mathcal{Z}_{\rm J}^{-1} \exp[-\beta(p^2+M^2)^{1/2}], \qquad \mathcal{Z}_{\rm J} = 2M \, K_1(\beta M). \tag{4.47}$$

[15] In principle, higher moments $\left\langle [\delta P(t)]^j \right\rangle_{\rm b}, j > 1$, can be calculated in a similar manner, but then one has to specify the j-particle heat bath PDF $f_{\rm b}^j$.

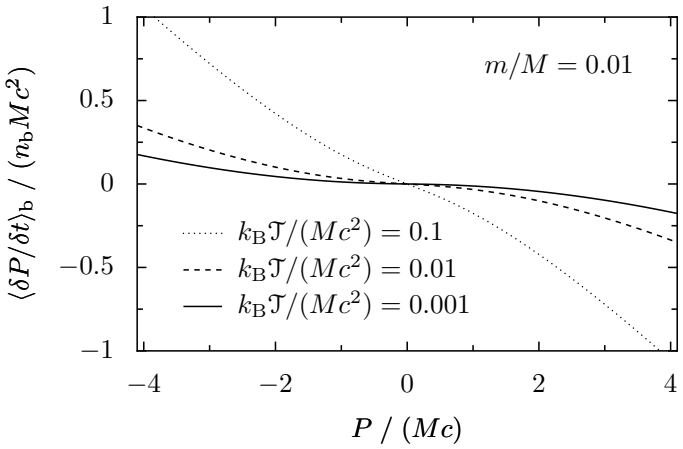


Figure 4.3: Relativistic binary collision model. Mean drift force $g(P) := \langle \delta P(t)/\delta t \rangle_{\mathrm{b}}$ numerically evaluated from Eq. (4.43) for different values $k_{\mathrm{B}}\mathcal{T}$, with $n_{\mathrm{b}} = N/L$ denoting the number density of the heat bath particles.

Hence, in order for Eq. (4.46) to yield the correct stationary distribution, the functions α and D must be coupled by the relativistic Einstein relation (4.2f), reading

$$D(p) = \beta^{-1}\alpha(p)\, E(p), \tag{4.48}$$

where $E = (p^2 + M^2)^{1/2}$. In order to determine the function α, we demand that the Langevin equation yields the same mean drift force g as the collision model, i.e.,

$$\left\langle \frac{\mathrm{d}P(t)}{\mathrm{d}t} \,\middle|\, P(t) = p \right\rangle \stackrel{!}{=} \left\langle \frac{\delta P(t)}{\delta t} \,\middle|\, P(t) = p \right\rangle_{\mathrm{b}}. \tag{4.49}$$

For the post-point (backward-Ito) Langevin equation (4.46) we know that [cf. Eq. (C.25)]

$$\left\langle \frac{\mathrm{d}P(t)}{\mathrm{d}t} \,\middle|\, P(t) = p \right\rangle = -\alpha(p)\, p + \frac{\mathrm{d}}{\mathrm{d}p} D(p). \tag{4.50}$$

Thus, by means of the Einstein relation (4.48), the lhs. of Eq. (4.49) is given by

$$\left\langle \frac{\mathrm{d}P(t)}{\mathrm{d}t} \,\middle|\, P(t) = p \right\rangle = -\alpha(p)\, p + \beta^{-1}\frac{\mathrm{d}}{\mathrm{d}p}[\alpha(p)\, E(p)], \tag{4.51}$$

and the condition (4.49) becomes equivalent to the differential equation [cf. Eq. (2.59)]

$$-\alpha(p)\, p + \beta^{-1}\frac{\mathrm{d}}{\mathrm{d}p}[\alpha(p)\, E(p)] = g(p). \tag{4.52}$$

In the case of the collision model, where the function $g(p)$ is not exactly known, one could, e.g., try to fit $g(p)$ by a simple analytic expression and, subsequently, use this approximation in Eq. (4.52).[16]

[16] Alternatively, one can also try to derive a Fokker-Planck equation from, e.g., a relativistic Boltzmann equation [278, 294].

Chapter 5

Non-Markovian relativistic diffusion

The preceding chapter was dedicated to relativistic Brownian motions in phase space. In the remainder we will discuss relativistic diffusion models in Minkowski space-time, i.e., continuous relativistic processes that do not explicitly depend on the momentum coordinate. On the one hand, such space-time processes may be constructed, e.g., from a Brownian motion processes in phase space by integrating out the momentum coordinates. As a result of this averaging procedure, the reduced process for the position coordinate will be non-Markovian. Alternatively, one can try to derive or postulate a relativistic diffusion equation and/or diffusion propagators by means of microscopic models [140, 375, 377, 381] or plausibility considerations [15]. Regardless of the approach adopted, in order to comply with the principles of special relativity, the resulting space-time process must be non-Markovian, as rigorously proven by Dudley (Theorem 11.3 in [341]) and Hakim (Proposition 2 in [346]). Put differently, any relativistic generalization of the nonrelativistic diffusion equation (1.1) with constant coefficients should be of at least second order in the time coordinate.

The construction and analysis of relativistic diffusion models in Minkowski space-time poses an interesting problem in its own right. In addition, the investigation of these processes becomes relevant in view of potential analogies with relativistic quantum theory [329, 398], similar to the analogy between Schrödinger's equation and the diffusion equation (1.1) in the nonrelativistic case [474, 475]. The present section intends to provide an overview over classical relativistic diffusion models that have been discussed in the literature [15, 329, 345, 375–377, 381, 390–392, 395, 396]. For this purpose, we first recall basic properties of the Wiener (Gaussian) process, which constitutes the standard paradigm for nonrelativistic diffusions in position space (Section 5.1). Subsequently, relativistic generalizations of the nonrelativistic diffusion equation (1.1) and/or the nonrelativistic Gaussian diffusion propagator will be discussed [15].

5.1 Reminder: nonrelativistic diffusion equation

We start by briefly recalling a few facts about the nonrelativistic standard diffusion equation (1.1), reading [220, 407, 475]

$$\frac{\partial}{\partial t}\varrho(t,x) = \mathcal{D}\,\nabla^2\varrho(t,x), \qquad t \geq 0, \tag{5.1}$$

where $\mathcal{D} > 0$ is the spatial diffusion constant, and $\nabla^2 = \partial^2/\partial x^2$ in the 1D case. It is well-known for a long time that the diffusion equation (5.1) is in conflict with the postulates of special relativity. A simple way of seeing this is to consider the propagator of Eq. (5.1), which for $d = 1$ space dimensions is given by

$$p(t,x|t_0,x_0) = \left[\frac{1}{4\pi\,\mathcal{D}(t-t_0)}\right]^{1/2} \exp\left[-\frac{(x-x_0)^2}{4\mathcal{D}(t-t_0)}\right], \qquad t > t_0. \tag{5.2}$$

The propagator (5.2) represents the solution of Eq. (5.1) for the initial condition

$$\varrho(t_0,x) = \delta(x-x_0).$$

That is, if $X(t)$ denotes the random path of a particle with fixed initial position $X(t_0) = x_0$, then $p(t,x|t_0,x_0)\mathrm{d}x$ gives the probability that the particle is found in the infinitesimal volume element $[x, x+\mathrm{d}x]$ at time $t > t_0$. As evident from Eq. (5.2), for each $t > t_0$ there is a small, but non-vanishing probability that the particle may be observed at distances $|x - x_0| > c(t-t_0)$, where $c = 1$ is the speed of light in natural units. The evolution of the nonrelativistic Gaussian PDF from Eq. (5.2) is depicted in Fig. 5.1.

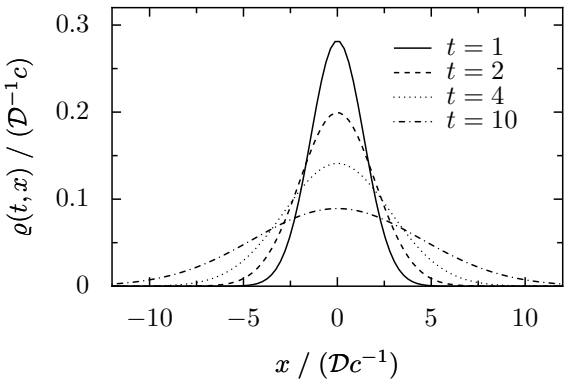


Figure 5.1: Spreading of the Gaussian PDF $\varrho(t,x) = p(t,x|0,0)$ from Eq. (5.2) at different times t, where t is measured in units of $\mathcal{D}/c^2$. At initial time $t = t_0 = 0$, the PDF corresponds to a δ-function centered at the origin.

We next summarize basic properties of Eqs. (5.1) and (5.2): Equation (5.1) is a linear parabolic partial differential equation. Due to the linearity, more general solutions may be constructed by superpositioning, i.e., by integrating the solution (5.2) over some given initial PDF $\varrho_0(x_0)$. Equation (5.1) describes a Markov process which means that the transition PDF (5.2) satisfies the Chapman-Kolmogoroff criterion

$$p(t,x|t_0,x_0) = \int_{\mathbb{R}} \mathrm{d}x_1 \; p(t,x|t_1,x_1)\, p(t_1,x_1|t_0,x_0) \tag{5.3}$$

for all $t_1 \in (t_0,t)$. The corresponding diffusion process $X(t)$ can be characterized in terms of the following SDE:

$$\mathrm{d}X(t) = (2\mathcal{D})^{1/2} * \mathrm{d}B(t), \qquad X(t_0) = x_0, \tag{5.4}$$

where $B(t)$ is a standard Wiener process as defined in Section 2.1.1. Formally, Eq. (5.4) may be obtained from the Langevin equation (2.3b) of the classical Ornstein-Uhlenbeck process as follows: First we rewrite Eq. (2.3b) as

$$\frac{\mathrm{d}V(t)}{M\alpha} \;=\; -V\mathrm{d}t + \left(\frac{2D}{M^2\alpha^2}\right)^{1/2} * \mathrm{d}B(t). \tag{5.5}$$

Upon letting $(M\alpha) \to \infty$ and $D \to \infty$ such that $\mathcal{D} = D/(\alpha M)^2$ remains constant, the lhs. of Eq. (5.5) should become negligible. Then, by making use of $\mathrm{d}X = V\mathrm{d}t$, Eq. (5.4) is recovered. This limiting procedure defines the so-called overdamped regime of the Ornstein-Uhlenbeck process. The mean square displacement of the overdamped process (5.4) is given by [220]

$$\begin{aligned} \big\langle [X(t)-X(t_0)]^2 \big\rangle \;&:=\; \int \mathrm{d}x\; (x-x_0)^2\, p(t,x|t_0,x_0) \\ &=\; 2\mathcal{D}\,(t-t_0), \end{aligned} \tag{5.6}$$

qualitatively similar to that of classical Ornstein-Uhlenbeck process; cf. Eq. (2.8). Finally, we note that the solution of Eq. (5.1) with initial condition

$$\varrho(t_0,x) \equiv \varrho_0(x), \tag{5.7}$$

can be expressed in the Feynman-Kac form [64, 66]

$$\varrho(t,x) = \big\langle \varrho_0\big(x + (2\mathcal{D})^{1/2} B(t)\big) \big\rangle \tag{5.8}$$

where $\langle\,\cdot\,\rangle$ indicates an average with respect to the standard Wiener process $B(t)$ with initial condition $B(t_0) = 0$. Equation (5.8) yields an efficient Monte-Carlo simulation scheme for computing the solutions of the diffusion equation (5.1).

5.2 Telegraph equation

The problem of constructing continuous diffusion models which, in contrast to Eqs. (5.1) and (5.2), avoid superluminal velocities, has attracted considerable interest over the past years [15, 329, 345, 375–377, 381, 390–392, 395, 396]. Nonetheless, it seems fair to say that a commonly accepted solution is still outstanding. Apart from the profound theoretical challenge of developing a consistent relativistic diffusion theory, there exist several practical applications including, e.g., the analysis of data from high energy collision experiments [393, 394, 406] or the diffusion of light through turbid media [383, 476, 477] and foams [384–386]. In this context, a frequently considered alternative to Eq. (5.1) is given by the telegraph equation [375, 377, 380, 388–391, 393–396]

$$\left(\tau_v \frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial t}\right) \varrho(t,x) = \mathcal{D}\,\nabla^2 \varrho(t,x). \tag{5.9}$$

Here, $\mathcal{D} > 0$ plays again the role of a diffusion constant, while $\tau_v > 0$ is an additional relaxation time parameter. Similar to Eq. (5.1), the telegraph equation (5.9) refers to a special frame where the background medium, causing the random motion of the diffusing test particle, is at rest (on average). The 'nonrelativistic limit' corresponds to letting $\tau_v \to 0$ in Eq. (5.9), which leads back to Eq. (5.1). For $\tau_v > 0$, Eq. (5.9) is a linear hyperbolic partial differential equation. Because of the second order time derivative in Eq. (5.9), one now also has to specify the first order time derivative of the initial distribution at time t_0. Considering particular initial conditions

$$\varrho(t_0,x) = \delta(x-x_0), \qquad \frac{\partial}{\partial t}\varrho(t_0,x) \equiv 0, \tag{5.10}$$

one finds that the corresponding solution of Eq. (5.9) is given by [375, 377]

$$\begin{aligned} p(t,x|t_0,x_0) \;=\; & \frac{e^{-(t-t_0)/(2\tau_v)}}{2} \left\{ \delta[|x-x_0| - v(t-t_0)] + \right. \\ & \left. \frac{\Theta(\xi^2)}{2\tau_v v} \left[I_0(\xi) + \frac{t}{2\tau_v} \frac{I_1(\xi)}{\xi} \right] \right\}. \end{aligned} \tag{5.11a}$$

Here, we have abbreviated

$$\xi := \frac{1}{2}\left[\left(\frac{t-t_0}{\tau_v}\right)^2 - \left(\frac{x-x_0}{\tau_v v}\right)^2 \right]^{1/2}, \qquad v := (\mathcal{D}/\tau_v)^{1/2}, \tag{5.11b}$$

and the modified Bessel functions of the first kind, $I_\nu(z)$, are defined by

$$I_\nu(z) := \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+\nu+1)\; k!} \left(\frac{z}{2}\right)^{2k+\nu}$$

with $\Gamma(z)$ denoting the Euler gamma function. According to our knowledge, the solution (5.11) was first obtained by Goldstein in 1938/1939. Actually, Goldstein derived the result (5.11) by considering the continuum limit of a persistent random walk model [379]; subsequently, he proved that this function satisfies the telegraph equation (5.9), cf. Section 8 of his paper [375].

The propagator (5.11) is characterized by two salient features:

a) As evident from the δ-function term, the solution exhibits two singular diffusion fronts traveling at absolute velocity $v := (\mathcal{D}/\tau_v)^{1/2}$ to the left and right, respectively;

b) due to the appearance of the Heaviside Θ-function, the solution is non-zero only within the region $|x - x_0| \leq v(t - t_0)$, i.e., upon fixing τ_v such that $v = c = 1$ the solution vanishes outside the light cone.

Thus, in contrast to the nonrelativistic propagator (5.2), Eqs. (5.11) define a relativistically acceptable diffusion model. Because of the second order time derivative, the telegraph equation (5.9) describes a *non-Markovian* process, in accordance with the aforementioned theorems of Dudley [341] and Hakim [346]. The non-Markovian character of the propagator (5.11) can also be proven directly by verifying that this solution does not fulfill the condition (5.3).

The linearity of Eq. (5.9) implies that more general solutions can be obtained by integrating the propagator (5.11) over some given initial distribution $\varrho_0(x_0)$. In principle, one may also construct other classes of solutions with $\partial\varrho(t_0, x)/\partial t \not\equiv 0$, e.g., by applying a Laplace-Fourier transformation [377, 478, 479] to Eq. (5.9). We note, however, that in order for the solution $\varrho(t, x)$ to remain normalized at all times $t > t_0$, it is required that

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, \frac{\partial}{\partial t} \varrho(t_0, x) \equiv 0, \tag{5.12a}$$

and

$$\frac{\partial}{\partial x} \varrho(t, \pm\infty) \equiv 0. \tag{5.12b}$$

In the case of Eq. (5.10), these conditions are automatically satisfied. Various solutions and extensions of the telegraph equation (5.9), including different types of boundary conditions, additional external sources, etc., have been discussed, e.g., by Goldstein [375], Masoliver et al. [478, 479], Foong and Kanno [480], Renardy [481], and Dorogovtsev [482].

Similar to the nonrelativistic diffusion equation (5.1), the telegraph equation (5.9) may be derived and/or motivated in many different ways. A detailed overview is given by Masoliver and Weiss [377], who discuss four different possibilities of deducing Eq. (5.9) from underlying models; see also Koide [395, 396]. During the past decades, the telegraph equation (5.9) has been used to describe a number of different phenomena. The applications include:

- *Transmission of electrical signals.* According to Masoliver and Weiss (5.9), the earliest derivation of the telegraph equation is based on a paper by William Thomson [483] (who later became Lord Kelvin), published in 1855. He considered the problem of how to transmit electrical signals without distortion, a question closely related to the design of the first transatlantic cable.

- *Continuum models of persistent diffusion.* It seems that the concept of persistent diffusion was first proposed in 1917/1922 by Fürth [378, 379] who aimed at describing the random motion of biological objects. Independently, a similar approach was suggested by Taylor [380] in an attempt to treat turbulent diffusion [376]. Fürth and Taylor considered discrete models, assuming that a particle moves with constant absolute velocity between neighboring lattice points. At each lattice point, the particle is either back-scattered or transmitted, with the transmission probability being larger than the back-scattering probability (*persistence*). A few decades later, in 1950, Goldstein [375] demonstrated for the 1D case[1] that, for a suitable choice of the transition probabilities, the continuum limit of this model leads to the telegraph equation. Hence, in contrast to the ordinary diffusion equation (5.1), the telegraph equation (5.9) relies on asymmetric transition probabilities, causing the non-vanishing probability concentration at the diffusion fronts. In more recent years, persistent diffusion models have been employed to describe the propagation of photons in thin slabs and foams [382–386].

- *Heat transport and propagation of heat waves.* In this case, the function $\varrho(t, x)$ in Eq. (5.9) is interpreted as a temperature field and the normalization condition is usually dropped. For a detailed account of the vast literature on heat waves we refer to the review article of Joseph and Preziosi [388, 389].

- *High energy ion collision experiments.* In recent years, the telegraph equation has been used to estimate the dissipation of net charge fluctuations, which may obliterate signals of QCD phase transitions in nuclear collisions [393, 394]. In this context, however, the coordinate x in Eq. (5.9) is interpreted as a rapidity variable.

Another interesting aspect of the telegraph equation is elucidated in a paper by Kac [376]. He observed that the solutions of Eq. (5.9) with initial conditions

$$\varrho(t_0, x) \equiv \varrho_0(x), \qquad \frac{\partial}{\partial t}\varrho(t_0, x) \equiv 0, \tag{5.13}$$

[1] Boguñá et al. [381] discuss persistent random walks in higher space dimensions.

may be expressed in the form[2]

$$\begin{aligned} \varrho(t,x) &= \frac{1}{2}\left\langle \varrho_0\left(x - v\int_{t_0}^{t} \mathrm{d}s\ (-1)^{N(s)}\right)\right\rangle + \\ &\quad \frac{1}{2}\left\langle \varrho_0\left(x + v\int_{t_0}^{t} \mathrm{d}s\ (-1)^{N(s)}\right)\right\rangle, \end{aligned} \tag{5.14}$$

where $v = (\mathcal{D}/\tau_v)^{1/2}$, and $\langle\,\cdot\,\rangle$ indicates an average with respect to the τ_v-parameterized Poisson process $N(t)$; i.e., for any given time $t > t_0$ we have

$$\mathrm{Prob}\{N(t) = k\} = \frac{e^{-(t-t_0)/(2\tau_v)}}{k!}\left(\frac{t-t_0}{2\tau_v}\right)^k, \qquad k = 0, 1, 2, \ldots; \tag{5.15a}$$

and for any finite sequence $t_0 < t_1 < \ldots < t_n$ the increments

$$N(t_1) - N(t_0),\ N(t_2) - N(t_1),\ \ldots,\ N(t_n) - N(t_{n-1}) \tag{5.15b}$$

are independent. Equation (5.14) is the direct counterpart of Feynman-Kac formula (5.8). Equations (5.14) and (5.15) together provide a very efficient Monte-Carlo simulation scheme for computing solutions of the telegraph equation (5.9). Moreover, the Poisson path integral representation (5.14) discloses an interesting correspondence between the free-particle Dirac equation [397] and the telegraph equation (5.9), which was first pointed out by Gaveau et al. [398] in 1984: The solutions of both equations may be linked by means of an analytic continuation. This connection is similar to the relation between the diffusion equation (5.1) and the free particle Schrödinger equation in the nonrelativistic case.[3] The crucial difference is given by the fact that the measures of the functional integration refer to different underlying processes, respectively.

However, the telegraph equation (5.9) is not the only possible relativistic generalization of the nonrelativistic diffusion equation (1.1) and, recently, there has been some controversy about its applicability and validity [390–392]. An early critical discussion of Eq. (5.9) in the context of relativistic heat transport was given by van Kampen [140] in 1970. Starting from a simple microscopic model, consisting of a cloud of material particles that exchange electromagnetic radiation, van Kampen derived an integral equation for the temperature of the particles as function of time and space. He then showed how the telegraph equation (5.9) can be recovered as an approximation to the more precise integral equation, but that the validity of this approximation breaks down in the vicinity of the diffusion fronts.

[2]The result (5.14) may be generalized to an arbitrary number of space dimensions; cf. pp. 500 in Kac's paper [376].

[3]For further reading about path integral representations of the Dirac propagator we refer to the papers of Ichinose [399,400], Jacobson and Schulman [401], Barut and Duru [402], and Gaveau and Schulman [403]; see also footnote 7 in Gaveau et al. [398] and problem 2-6, pp. 34-36 in Feynman and Hibbs [404].

Similarly, the singular diffusion fronts predicted by Eq. (5.11) represent a source of concern if one wishes to adopt the telegraph equation (5.9) as a model for particle transport in a random medium. While these singularities may be acceptable in the case of photon diffusion [382–386], they seem unrealistic for massive particles, because such fronts would imply that a finite fraction of particles carries a huge amount of kinetic energy (much larger than mc^2). In view of these shortcomings, it appears reasonable to explore other constructions of relativistic diffusion processes [15, 405]. In the next part we will propose a different approach [15] that may provide a viable alternative to the solutions of the telegraph equation.

5.3 Relativistic diffusion propagator

In principle, one can distinguish two different routes towards constructing relativistic diffusion processes: One can either try to find an acceptable relativistic diffusion equation, or one can focus directly on the structure of the diffusion propagator. In the present part we shall adopt the latter approach [15]. The basic idea is to rewrite the nonrelativistic diffusion propagator (5.2) in such a form that its relativistic generalized follows in a straightforward manner. This can be achieved be reexpressing Eq. (5.2) in terms of an integral-over-actions.

For this purpose, we consider a nonrelativistic particle traveling from the event $\bar{x}_0 = (t_0, x_0)$ to $\bar{x} = (t, x)$. We assume that the particle can experience multiple scatterings on its way, and that the velocity is approximately constant between two successive scattering events. Then the total action (per mass) required along the path is given by

$$a(\bar{x}|\bar{x}_0) = \frac{1}{2}\int_{t_0}^{t} \mathrm{d}t'\; v(t')^2, \tag{5.16}$$

where the velocity $v(t')$ is a piecewise constant function, satisfying

$$x = x_0 + \int_{t_0}^{t} \mathrm{d}t'\; v(t'). \tag{5.17}$$

Clearly, the nonrelativistic action (5.16) becomes minimal for the deterministic (direct) path, i.e., if the particle does *not* collide at all. In this case, it moves with constant velocity $v(t') \equiv (x - x_0)/(t - t_0)$ for all $t' \in [t_0, t]$, yielding the smallest possible action value

$$a_-(\bar{x}|\bar{x}_0) = \frac{(x - x_0)^2}{2(t - t_0)}. \tag{5.18}$$

On the other hand, to match the boundary conditions it is merely required that the mean velocity equals $(x - x_0)/(t - t_0)$. Consequently, in the nonrelativistic case, the absolute

velocity of a particle may become arbitrarily large during some intermediate time interval $[t', t''] \subset [t_0, t]$. Hence, the largest possible action value is $a_+(\bar{x}, \bar{x}_0) = +\infty$. These considerations put us in the position to rewrite the Wiener propagator (5.2) as an integral-over-actions:

$$p(\bar{x}|\bar{x}_0) \;\propto\; \int_{a_-(\bar{x}|\bar{x}_0)}^{a_+(\bar{x}|\bar{x}_0)} \mathrm{d}a \; \exp\!\left(-\frac{a}{2\mathcal{D}}\right), \tag{5.19a}$$

supplemented by the normalization condition

$$1 = \int \mathrm{d}x \; p(\bar{x}|\bar{x}_0). \tag{5.19b}$$

The representation (5.19) may be generalized to the relativistic case in a straightforward manner: One merely needs to insert the corresponding relativistic expressions into the boundaries of the integral (5.19a). A commonly considered relativistic generalization of Eq. (5.16), based on the particle's proper time, reads [6]

$$a = -\int_{t_0}^{t} \mathrm{d}t' \;\left[1 - v(t')^2\right]^{1/2}. \tag{5.20}$$

Analogous to the nonrelativistic case, the relativistic action (5.20) assumes its minimum a_- for the deterministic (direct) path from x_0 to x, characterized by a constant velocity $v(t') \equiv (x - x_0)/(t - t_0)$. One explicitly obtains

$$a_-(\bar{x}, \bar{x}_0) = -\left[(t - t_0)^2 - (x - x_0)^2\right]^{1/2}, \tag{5.21a}$$

i.e., a_- is the negative Minkowski distance of the two space-time events $\bar{x}_0$ and $\bar{x}$. The maximum action value is realized for particles moving at light speed, yielding $a_+ = 0$. Hence, the transition PDF for the relativistic generalization of the Wiener process reads

$$p(\bar{x}|\bar{x}_0) = \mathcal{N}^{-1} \;\left\{\exp\!\left[-\frac{a_-(\bar{x}, \bar{x}_0)}{2\mathcal{D}}\right] - 1\right\}, \tag{5.21b}$$

if $(x - x_0)^2 \leq (t - t_0)^2$, and $p(\bar{x}|\bar{x}_0) \equiv 0$ otherwise, with a_- determined by Eq. (5.21a).

The relativistic diffusion process defined by Eqs. (5.21) is *non-Markovian*, i.e., it does not fulfill Chapman-Kolmogoroff criterion (5.3). The functional form of the propagator (5.21b) remains the same for higher space dimensions $d > 1$; the corresponding normalization constants are given in the App. B. In contrast to the solution (5.11) of the telegraph equation, the propagator (5.21b) vanishes continuously at the diffusion fronts. Figure 5.2 depicts the PDF $\varrho(t, x) = p(t, x|0, 0)$ of the diffusion process (5.21) for the one-dimensional case $d = 1$ at different times t. The corresponding mean square displacement is plotted in Fig. 5.3 (dashed curve).

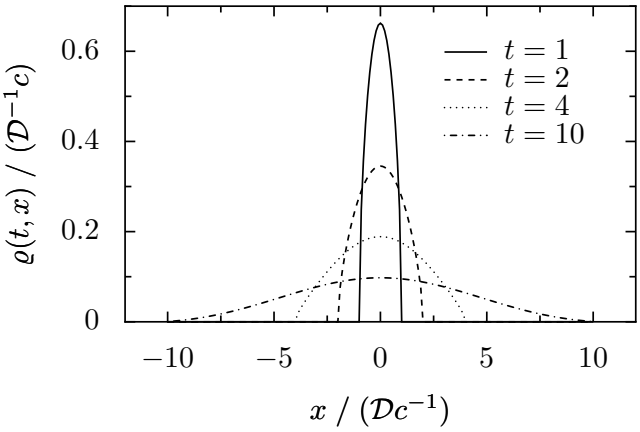


Figure 5.2: Transition PDF $\varrho(t,x) = p(t,x|0,0)$ for the one-dimensional ($d = 1$) relativistic diffusion process (5.21) at different times t (measured in units of $\mathcal{D}/c^2$). At time $t = t_0 = 0$, the function $\varrho(t,x)$ reduces to a δ-function centered at $x_0 = 0$. In contrast to the nonrelativistic diffusion propagator, cf. Fig. 5.1, the PDF (5.21) vanishes outside of the light cone.

It is interesting to note that the PDF (5.19) is a special case of a larger class of diffusion processes, defined by

$$p_w(\bar{x}|\bar{x}_0) \;=\; \mathcal{N}_w^{-1} \int_{a_-(\bar{x}|\bar{x}_0)}^{a_+(\bar{x}|\bar{x}_0)} \mathrm{d}a\; w(a), \tag{5.22}$$

where $w(a) \geq 0$ is a weighting function, and $\mathcal{N}_w$ the time-dependent normalization constant. In particular, Eq. (5.22) may be viewed as a path integral definition in the following sense: Physically permissible paths from $\bar{x}_0$ to $\bar{x}$ have action values (per mass) a in the range $[a_-, a_+]$. Grouping the different paths together according to their action values, one may assign to each such class of paths, denoted by $\mathcal{C}(a;\bar{x},\bar{x}_0)$, the statistical weight $w(a)$. The integral (5.22) can then be read as an integral over the equivalence classes $\mathcal{C}(a;\bar{x},\bar{x}_0)$ and their respective weights $w(a)$. The nonrelativistic Wiener process corresponds to the specific choice $w(a) = \exp[-a/(2\mathcal{D})]$; hence, it appears natural to define the relativistic generalization by using the same weighting function. It is, however, worth mentioning that a very large class of functions $w(a)$ yields an asymptotic growth of the spatial mean square displacement that is proportional to t, corresponding to 'ordinary' diffusion. Moreover, Eq. (5.22) can also be used to describe super-diffusion or sub-diffusion processes [105, 106, 484], whose asymptotic mean square displacements grow as t^α, $\alpha \neq 1$.[4]

[4]This can be achieved, e.g., by choosing the integral boundaries as $\tilde{a}_- = (x - x_0)^2/(t - t_0)^\alpha$, $\alpha \neq 1$ and $a_+ = \infty$, but then the variable a may not be interpreted as a conventional action anymore.

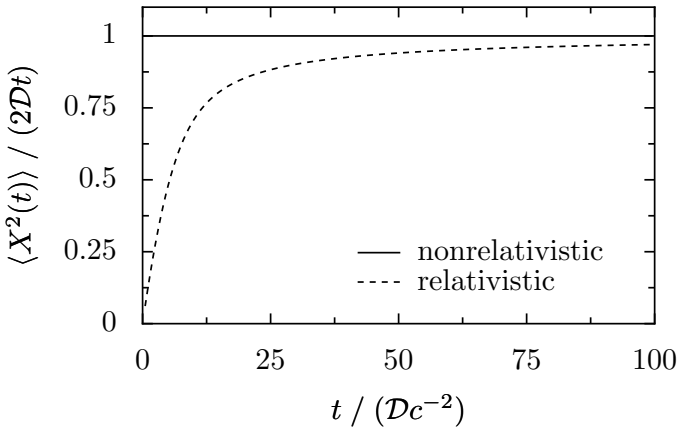


Figure 5.3: Comparison of the mean square displacements $\langle X^2(t)\rangle$, divided by $2\mathcal{D}t$, for the one-dimensional ($d = 1$) nonrelativistic Wiener process (5.2) and its relativistic generalization from Eq. (5.21) with initial condition $(t_0, x_0) = (0, 0)$.

Chapter 6

Summary and outlook

In this thesis, we have investigated how the concepts of Brownian motion and diffusion [4] may be incorporated into the framework of special relativity [2, 3]. As the starting point, we summarized in Chapter 2 the foundations of the nonrelativistic Langevin theory [31,32] of Brownian motions. Langevin equations are stochastic differential equations (SDEs), and provide a simplified description of the complex microscopic interactions between a Brownian particle and its environment [64,65]. Nonrelativistic stochastic processes as, e.g., the classical Ornstein-Uhlenbeck process [36, 39] can be used as guidance for constructing relativistic Langevin equations. In particular, these nonrelativistic processes should also be recovered from the relativistic theory in the limit of an infinite light speed $c \to \infty$. Since, in the past, nonrelativistic SDEs have proven to be very useful for describing a wide range of different phenomena [66, 79, 336], relativistic Langevin equations can be expected to play a similarly successful role in the modeling of thermalization and relaxation processes in relativistic and astrophysical system [312–314, 337, 338, 471].

When adopting a phenomenological or axiomatic Langevin equation approach, the fluctuation-dissipation relations of the friction and noise coefficients have to be chosen such that the correct equilibrium distributions are recovered in the stationary state. For example, in the nonrelativistic case it is commonly accepted that the stationary velocity distribution of a free[1] Brownian particle in a thermally equilibrated environment (heat bath) is given by a Maxwell distribution [168]. As well-known, Maxwell's distribution is in conflict[2] with special relativity and, therefore, has to be replaced in a relativistic theory. In recent years, there has been considerable debate in the literature about the correct generalization of Maxwell's distribution in special relativity [14, 206, 208, 209, 211, 215–217]. In order to elucidate the differences between two of the most commonly considered candidate distributions [14, 167, 211], we considered a maximum principle for the relative entropy [16],

[1] By 'free' we mean the absence of external fields.

[2] The Maxwell distribution, representing a Gaussian velocity distribution, assigns a non-vanishing probability to superluminal velocities.

combined with group-theoretical considerations (Chapter 3). Thereby, it could be shown that the two candidate distributions correspond to different underlying reference measures (i.e., different relative entropies), which are characterized by different symmetries, respectively.

Knowledge of the correct relativistic equilibrium velocity distributions is essential for the reliable interpretation of experiments in high energy physics [312–314] and astrophysics [317, 328, 338], as well as for the derivation of relativistic Langevin equations from microscopic models [13]. In order to identify the correct equilibrium distribution 'experimentally', we performed fully relativistic one-dimensional (1D) molecular dynamics simulations in collaboration with David Cubero and Jesús Casado-Pascual [17]. Our computer experiments clearly favor the Jüttner distribution [167, 169, 198] as the correct relativistic one-particle equilibrium velocity distribution.

Subsequent to the discussion of relativistic equilibrium thermostatistics in Chapter 3, the Langevin theory of relativistic Brownian motions in phase space was developed in Chapter 4. After clarifying the general conceptual and mathematical foundations, it was demonstrated that the different relativistic Brownian processes proposed in Refs. [18, 19] and [10, 11], respectively, can be viewed as special limit cases within a class of more general relativistic Langevin equations. The comparison of analytical and numerical results for different example processes [23, 24] shows that at moderate-to-high temperatures the asymptotic diffusion constant can significantly vary for different friction models (Section 4.1.3). For realistic systems, the exact functional shape (i.e., energy dependence) of the friction function α is determined by the microscopic interactions. This implies that simultaneous measurements of the temperature and the diffusion constants can reveal information about the structure of the underlying microscopic forces.

The transformation behavior of Langevin equations under Lorentz transformations was discussed in Section 4.2. By considering a Lorentz transformation of the corresponding Fokker-Planck equation, it was explicitly demonstrated for the 1D case that a Langevin process defined in the lab frame transforms into a Langevin process in the moving frame. Furthermore, to better understand the applicability of relativistic Langevin equations, we proposed and analyzed a simple 1D binary collision model of relativistic Brownian motion [13]. This model helps to clarify the approximations that must be made in order to derive a relativistic Langevin equation from an underlying microscopic dynamics (Section 4.3). These investigations may provide useful guidance for future studies, e.g., with regard to the derivation of relativistic Langevin-type equations from more complex microscopic models; cf. remarks below.

Finally, in the last part of this thesis (Chapter 5), we discussed relativistic generalizations of the nonrelativistic Wiener diffusion propagator in Minkowski space-time. The Wiener propagator represents the fundamental solution of the nonrelativistic diffusion equation and describes a Markovian diffusion process in position space. This process violates the basic

principles of relativity as particles are permitted to diffuse at superluminal speed. Since nontrivial relativistic Markov processes in Minkowski space-time (i.e., time and position space) do not exist [339, 341, 346], relativistic generalizations of the Wiener process must necessarily be non-Markovian. A commonly considered alternative to the nonrelativistic diffusion equation is the telegraph equation [377]. In contrast to the classical diffusion equation, the telegraph equation is a second order partial differential equation with respect to the time coordinate and, thus, describes a non-Markovian process. However, the solutions of the telegraph equation suffer from divergences on the light cone [377]. Attempting to overcome this deficiency, we propose in Section 5.3 a novel relativistic generalization of the Wiener diffusion propagator [15]. The functional form of the propagator was deduced from an integral-over-actions representation of the nonrelativistic propagator. The resulting relativistic diffusion propagator is continuous, vanishes outside the light cone, and converges to the classical Wiener process in the asymptotic limit case. The proposed propagator could, therefore, provide a viable alternative to the solutions of the telegraph equation.

We conclude our discussion by summarizing problems which, in our opinion, deserve further consideration in the future:

- *Microscopic models.* The 1D binary collision model from Sec. 4.3 appears to be the simplest example for deriving a relativistic Langevin-type equation from an underlying microscopic model. Future investigations should focus on constructing relativistic Langevin equations from more precise particle-field interaction models; e.g., one could consider the motion of a classical relativistic point particle in quasi-static external random fields. If successful, this approach would yield more precise noise and friction models for relativistic systems.[3] Moreover, this would contribute to clarifying the applicability of stochastic differential equations to relativistic problems. A useful starting point for future research in this direction could be the work of Blanco et al. [327], Johnson and Hu [491–493] and Galley et al. [494], who proposed to model the interaction between quantum test particles and vacuum fluctuations of quantum fields by means of effective Langevin equations.

- *Relativistic processes driven by non-Gaussian noise.* Most of the relativistic stochastic differential equations discussed in the literature thus far [10, 11, 13, 18–22, 329, 331, 333, 334] are driven by Brownian motion (Wiener) processes that couple to the momentum coordinates. Therefore, it would be interesting to consider other driving processes as well (e.g., Poisson or Lévy noise) and to compare with the results of the corresponding nonrelativistic equations [73, 445]. On the one hand, the properties of such generalized relativistic processes deserve to be studied from a purely mathematical point of view, e.g., with regard to potential modifications of anomalous diffusion

[3] In this context, technical and conceptual challenges are likely to arise when considering the energy loss of the particle due to radiation emission [485–490].

effects. On the other hand, it would be desirable to identify classes of physical systems that can be described by these processes. For example, it seems likely that that the quasi-random particle acceleration [495] in various astrophysical systems may be efficiently modeled by non-Gaussian driving processes.

- *Relativistic fluctuation theorems.* Fluctuation theorems are mathematical relations that may be used to deduce certain thermodynamic properties of equilibrium systems by measuring suitable averages from different realizations of a non-equilibrium process [496–500]. During the past two decades, numerous fluctuation theorems have been established for various closed and open nonrelativistic systems. Both from a theoretical and practical point of view, it would be most interesting to generalize these results to the relativistic case. Recently, a first step in this direction was made by Fingerle [470], who derived a relativistic fluctuation theorem for the special relativistic Brownian motion process proposed in [10]. It will be worthwhile to extend these investigations to other, more general relativistic processes (see also Cleuren et al. [218]).

- *Relativistic diffusion models in Minkowski space.* The above problems refer to stochastic processes in relativistic phase space. Alternatively, one may focus on constructing and analyzing novel types of relativistic diffusion processes in Minkowski space-time, similar to those discussed in Sec. 5. One particularly important issue in this context concerns the existence of reasonable path integral representations for the propagators of such processes. The latter question is closely related to the problem of finding path integral formulations of relativistic quantum propagators [399, 400, 402, 404].

- *Relativistic quantum Brownian motions.* Another potential generalization of the above ideas concerns the construction of relativistic quantum Brownian motion processes. Recent efforts in this direction include the aforementioned papers by Johnson and Hu [491–493] and Galley et al. [494], who considered the motion of quasi-particles in fluctuating quantum fields. Similar concepts have also been studied within the theory of stochastic semiclassical gravity, where the gravitational field plays the role of a 'Brownian particle' with the vacuum fluctuations of quantum fields forming a stochastic environment ('bath'); for details we refer to the recent review by Hu and Verdaguer [501]. Another promising starting point may be work the work of Pechukas [95], and Tsonchev and Pechukas [96], who developed a simple elastic collision model of nonrelativistic quantum Brownian motions. In this context, we also mention the work of Breuer and Petruccine [502–504], who proposed a Langevin equation approach to describe quantum state diffusion in the framework of special relativity, see also Diosi [505].

- *Extensions to general relativity.* Last but not least, notwithstanding recent progress [372–374, 470, 472, 506–509], the generalization of stochastic concepts and their applications within the framework of general relativity offers many interesting challenges for the future.

Appendix A

Special relativity (basics)

This appendix summarizes basic definitions and a few properties of Lorentz transformations. For a more detailed introduction to special relativity we refer to Weinberg [6] or Sexl and Urbantke [8].

A.1 Notation and definitions

In special relativity, an inertial frame Σ corresponds to a global Cartesian space-time coordinate system. A space-time event is labeled by a $(1+d)$-dimensional coordinate tupel $\bar{x} = (x^\alpha) = (ct, \boldsymbol{x}) = (t, x^1, \ldots, x^d)$ in Σ, where d is the number of space dimensions and, adopting natural units, the speed of light $c = 1$. Upper and lower Greek indices $\alpha, \beta, \ldots$ take values $0, 1, \ldots, d$, and we use Latin indices $i, k, \ldots \in \{1, \ldots, d\}$ for the spatial components. Vectors with upper indices are called contravariant.

With respect to the Cartesian coordinate frame Σ, the components $\eta_{\alpha\beta}$ of the metric tensor of flat Minkowski space-time are defined by [6]

$$\eta_{\alpha\beta} = \begin{cases} -1 & \alpha = \beta = 0 \\ +1 & \alpha = \beta = 1, \ldots, d \\ 0 & \alpha \neq \beta \end{cases} \tag{A.1}$$

By definition, the components of the covariant vector (x_α) are obtained by contracting the contravariant vector (x^α) with $\eta_{\alpha\beta}$, i.e.,[1]

$$x_\alpha := \sum_{\beta=0}^{d} \eta_{\alpha\beta} x^\beta =: \eta_{\alpha\beta} x^\beta, \tag{A.2}$$

[1]Sometimes, it is convenient to interpret the contravariant vectors (a^α) as column vectors and the corresponding covariant vectors (a_α) as row vectors.

yielding explicitly

$$(x_\alpha) = (-t, \boldsymbol{x}). \tag{A.3}$$

The rhs. of Eqs. (A.2) defines the Einstein's summation convention. The vectors (x^α) and (x_α) will be called four-vectors, regardless of the number of space dimensions. The corresponding four-vector gradients are defined by

$$(\partial_\alpha) := \left(\frac{\partial}{\partial x^\alpha}\right) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^d}\right) = \left(\frac{\partial}{\partial t}, \nabla\right), \tag{A.4a}$$

$$(\partial^\alpha) := \left(\frac{\partial}{\partial x_\alpha}\right) = \left(-\frac{\partial}{\partial t}, \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^d}\right) = \left(-\frac{\partial}{\partial t}, \nabla\right). \tag{A.4b}$$

The components $\eta^{\alpha\beta}$ of the inverse metric tensor are determined by the condition

$$x^\alpha \stackrel{!}{=} \eta^{\alpha\beta} x_\beta = \eta^{\alpha\beta}\eta_{\beta\gamma} x^\gamma \qquad \forall\ (x^\alpha), \tag{A.5a}$$

or, equivalently, by

$$\eta^{\alpha\beta}\eta_{\beta\gamma} \stackrel{!}{=} \delta^\alpha{}_\gamma, \tag{A.5b}$$

where $\delta^\alpha{}_\gamma$ is the Kronecker δ-symbol, yielding

$$\eta^{\alpha\beta} = \eta_{\alpha\beta}. \tag{A.6}$$

The Minkowski space-time distance between two events $\bar{x}_A = (x^\alpha_A) = (t_A, \boldsymbol{x}_A)$ and $\bar{x}_B = (x^\alpha_B) = (t_B, \boldsymbol{x}_B)$ is defined by

$$\begin{aligned} d(\bar{x}_A, \bar{x}_B)^2 &:= \eta_{\alpha\beta}(x^\alpha_A - x^\alpha_B)(x^\beta_A - x^\beta_B) \\ &= -(t_A - t_B)^2 + (\boldsymbol{x}_A - \boldsymbol{x}_B)^2. \end{aligned} \tag{A.7}$$

By definition, the separation of two events is

- time-like, if $d(\bar{x}_A, \bar{x}_B)^2 < 0$;
- light-like, if $d(\bar{x}_A, \bar{x}_B)^2 = 0$;
- space-like, if $d(\bar{x}_A, \bar{x}_B)^2 > 0$.

In special relativity, events with time-like separation can be causally connected by (a series of) signals travelling slower or equal to the speed of light. Events with light-like separation can be causally related only by undisturbed signals travelling at the speed of light. Events with space-like separation are causally disconnected.

The classical motion of a massive particle through space-time corresponds to a sufficiently smooth, time-like curve in Σ, referred to as world-line. In the vicinity of any point (event) on the particle's world-line, an infinitesimal proper time differential can be defined by

$$\mathrm{d}\tau := (-\eta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta)^{1/2} = (\mathrm{d}t^2 - \mathrm{d}\boldsymbol{x}^2)^{1/2} = \mathrm{d}t\,(1-\boldsymbol{v}^2)^{1/2}, \tag{A.8}$$

where $\boldsymbol{v} := \mathrm{d}\boldsymbol{x}/\mathrm{d}t$ is the particle velocity in Σ. According to special relativity, $\mathrm{d}\tau$ is the time interval measured by an intrinsic clock, comoving with the particle, while $\mathrm{d}t$ is the coordinate time interval measured by a clock at rest in Σ.

The four-vector velocity (u^α) of a massive particle is defined as the derivative of the world-line with respect to proper time,

$$u^\alpha := \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}, \tag{A.9a}$$

and satisfies, by construction,

$$u_\alpha u^\alpha = -1. \tag{A.9b}$$

The momentum four-vector $(p^\alpha) = (p^0, p^1, \ldots, p^d) = (E, \boldsymbol{p})$ is defined by

$$p^\alpha := m u^\alpha \qquad \Rightarrow \qquad p_\alpha p^\alpha = -m^2, \tag{A.10}$$

where $m > 0$ is the rest mass of the particle. Upon comparing with (A.8), one finds for a particle with velocity $\boldsymbol{v}$ in Σ

$$p^0 = E = m(1-\boldsymbol{v}^2)^{-1/2}\,, \qquad\qquad \boldsymbol{p} = E\boldsymbol{v} = m\boldsymbol{v}(1-\boldsymbol{v}^2)^{-1/2}. \tag{A.11}$$

A.2 Lorentz-Poincaré transformations

In special relativity, a Lorentz-Poincaré transformation (LPT) describes the transition from an inertial frame Σ to another inertial frame Σ'. Mathematically, an LPT is a linear transformation of the form

$$x'^\alpha = \Lambda^\alpha{}_\beta x^\beta + a^\alpha, \tag{A.12a}$$

mapping the 'old' Σ-space-time-coordinates x^α onto the 'new' Σ'-coordinates x'^α. The constant four-vector a^α shifts the origins of time and space, while the constant Lorentz matrix $(\Lambda^\alpha{}_\beta)$ may account for a spatial rotation, a change of orientation and/or a relative velocity between the two frames Σ and Σ', cf. Chapter 1 of Weinberg's book [6]. The matrix components $\Lambda^\alpha{}_\beta$ are determined by the condition

$$d(\bar{x}'_A, \bar{x}'_B)^2 \stackrel{!}{=} d(\bar{x}_A, \bar{x}_B)^2 \qquad \Leftrightarrow \qquad \Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta \eta_{\alpha\beta} \stackrel{!}{=} \eta_{\gamma\delta}. \tag{A.12b}$$

The condition (A.12b) means that causal relations remain preserved during transitions between inertial systems.

The LPTs (A.12) form a group.[2] Of particular interest for our purpose, is the subgroup of the *proper* LPTs, defined by imposing the additional constraints

$$\Lambda^0{}_0 \geq 1 \,, \qquad\qquad \det(\Lambda^\alpha{}_\beta) = +1. \tag{A.13}$$

The requirements (A.13) exclude time reversal and space inversion. Examples are pure rotations

$$\Lambda^0{}_0 = 1 \,, \qquad\qquad \Lambda^i{}_0 = \Lambda^0{}_i = 0 \,, \qquad\qquad \Lambda^i{}_j = R_{ij}, \tag{A.14}$$

where (R_{ij}) is a rotation matrix [i.e., $\det(R_{ij}) = 1$ and $R_{ij}R_{kj} = \delta_{ij}$], and Lorentz boosts [6]

$$\Lambda^0{}_0 = \gamma, \qquad \Lambda^i{}_0 = \Lambda^0{}_i = \gamma w^i, \qquad \Lambda^i{}_j = \delta^i{}_j + w^i w^j \frac{(\gamma - 1)}{\boldsymbol{w}^2}. \tag{A.15}$$

with velocity $\boldsymbol{w} = (w^1, \ldots, w^d)$ and Lorentz factor

$$\gamma := (1 - \boldsymbol{w}^2)^{-1/2}. \tag{A.16}$$

To illustrate the effect of a boost, we first consider a particle that rests at the spatial origin of Σ and, therefore, being described by $(x^\alpha) = (t, \mathbf{0})$ in Σ. By applying the Lorentz boost (A.15) to $(x^\alpha) = (t, \mathbf{0})$, we find

$$x'^0 = \Lambda^0{}_0 x^0 = \gamma t = t' \,, \qquad\qquad x'^i = \Lambda^i{}_0 x^0 = \gamma w^i t = w^i t', \tag{A.17}$$

which means that the particle moves at constant velocity $\boldsymbol{w}$ through Σ'. Similarly, we may consider a particle that moves at velocity $-\boldsymbol{w}$ through Σ. If the particle at Σ-time $t = 0$ was located at $\boldsymbol{x} = 0$ then it is described by the world-line $(x^\alpha) = (t, -t\boldsymbol{w})$ in Σ. By applying the Lorentz boost (A.15) to $(x^\alpha) = (t, -t\boldsymbol{w})$, we obtain

$$x'^0 = \gamma t - \gamma w^i w^i t = t/\gamma = t' \tag{A.18a}$$

$$x'^i = \gamma w^i t + \left[\delta^i{}_j + w^i w^j \frac{(\gamma - 1)}{\boldsymbol{w}^2}\right](-t w^j) = 0, \tag{A.18b}$$

i.e., the particle rests in the spatial origin of Σ'.

From Eq. (A.12) and the definition (A.10) of the four-momentum, one finds the momentum transformation law

$$p'^\alpha = \Lambda^\alpha{}_\beta p^\beta. \tag{A.19}$$

[2] A detailed discussion can be found in [6, 8, 9].

Combining Eqs. (A.19) and (A.12b), one can verify the mass-shell condition

$$m^2 = E^2 - \boldsymbol{p}^2 = E'^2 - \boldsymbol{p}'^2 = m'^2, \tag{A.20}$$

which means that the rest mass m is a Lorentz invariant. In particular, the mass shell condition (A.20) implies that Eq. (A.19) is equivalent to the nonlinear restricted transformation $\boldsymbol{p} \mapsto \boldsymbol{p}'$, given by

$$p'^i(\boldsymbol{p}) = \Lambda^i{}_0(m^2 + \boldsymbol{p}^2)^{1/2} + \Lambda^i{}_j p^j, \tag{A.21a}$$

and we find

$$\frac{\partial p'^i}{\partial p^k} = \Lambda^i{}_0 \frac{p_k}{(m^2 + \boldsymbol{p}^2)^{1/2}} + \Lambda^i{}_k. \tag{A.21b}$$

Appendix B

Normalization constants

B.1 Jüttner function

The normalization constant $\mathcal{Z}_d$ of the d-dimensional Jüttner distribution

$$\phi_{\mathrm{J}}(\boldsymbol{p}) = (\mathcal{Z}_d)^{-1} \exp[-\beta E(\boldsymbol{p})], \tag{B.1}$$

with $E(\boldsymbol{p}) = (\boldsymbol{p}^2 + m^2)^{1/2}$, is given by the integral

$$\mathcal{Z}_d = \int \mathrm{d}^d\boldsymbol{p}\ \exp[-\beta(\boldsymbol{p}^2 + m^2)^{1/2}]. \tag{B.2}$$

Using spherical momentum coordinates, one evaluates

$$\mathcal{Z}_1 = 2m\, K_1(\beta m), \tag{B.3a}$$

$$\mathcal{Z}_3 = 4\pi m^3\, \frac{K_2(\beta m)}{\beta m}, \tag{B.3b}$$

with $K_n(z)$ denoting the modified Bessel functions of the second kind [466]. The energy mean values are obtained by logarithmic differentiation,

$$\langle E \rangle_d = -\frac{\partial}{\partial\beta} \ln \mathcal{Z}_d, \tag{B.4}$$

and, upon inserting Eqs. (B.3), one finds

$$\langle E \rangle_1 = m\, \frac{K_0(\beta m) + K_2(\beta m)}{2K_1(\beta m)}, \tag{B.5a}$$

$$\langle E \rangle_3 = \frac{3}{\beta} + m\, \frac{K_1(\beta m)}{K_2(\beta m)}. \tag{B.5b}$$

B.2 Diffusion propagator

We wish to express the normalization constant $\mathcal{N}_d$ of the d-dimensional diffusion propagator [from Section 5.3]

$$p(\bar{x}|\bar{x}_0) = \mathcal{N}_d^{-1} \left\{ \exp\left[-\frac{a_-(\bar{x}, \bar{x}_0)}{2\mathcal{D}} \right] - 1 \right\}, \qquad \bar{x} = (t, \boldsymbol{x}) \tag{B.6}$$

in terms of modified Bessel functions of the first kind and modified Struve functions [466]. Introducing $\boldsymbol{z} := \boldsymbol{x} - \boldsymbol{x}_0$ and $u := t - t_0$, we have to calculate

$$\mathcal{N}_d \;=\; \int_{\mathbb{R}^d} \mathrm{d}^d \boldsymbol{z}\; \Theta(u - |\boldsymbol{z}|) \left\{ \exp\left[\frac{(u^2 - \boldsymbol{z}^2)^{1/2}}{2\mathcal{D}} \right] - 1 \right\}.$$

Using spherical coordinates, we can rewrite this as

$$\mathcal{N}_d = O_d \int_0^u \mathrm{d}|\boldsymbol{z}|\; |\boldsymbol{z}|^{d-1} \left\{ \exp\left[\frac{(u^2 - |\boldsymbol{z}|^2)^{1/2}}{2\mathcal{D}} \right] - 1 \right\}, \tag{B.7}$$

where $O_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the d-dimensional unit-sphere. It is convenient to split the integral (B.7) in the form

$$\mathcal{N}_d \;=\; \mathcal{N}'_d - \frac{u^d}{d} O_d, \tag{B.8}$$

where

$$\mathcal{N}'_d \;=\; O_d \int_0^u \mathrm{d}|\boldsymbol{z}|\; |\boldsymbol{z}|^{d-1}\; \exp\left[\frac{(u^2 - |\boldsymbol{z}|^2)^{1/2}}{2\mathcal{D}} \right], \tag{B.9}$$

Next we substitute $|\boldsymbol{z}| = u\, \sin\phi$, where $\phi \in [0, \pi/2]$. Then Eq. (B.9) takes the form

$$\mathcal{N}'_d \;=\; u^d\, O_d \int_0^{\pi/2} \mathrm{d}\phi\; \cos\phi\; \sin^{d-1}\phi\; \exp\left(\frac{u\, \cos\phi}{2\mathcal{D}} \right).$$

Based on this integral representation, $\mathcal{N}'_d$ can be expressed in terms of modified Bessel functions of the first kind I_n, and modified Struve functions L_k [466], and one finds

$$\mathcal{N}'_1 \;=\; u\, \pi\, \left[I_1(\chi) + L_{-1}(\chi) \right], \tag{B.10a}$$

$$\mathcal{N}'_2 \;=\; u^2\, \frac{2\pi}{\chi^2}\, \left[1 + (\chi - 1)\, \exp(\chi) \right], \tag{B.10b}$$

$$\mathcal{N}'_3 \;=\; u^3\, \frac{2\pi^2}{\chi^2}\, \left\{ \chi\, \left[I_2(\chi) + L_0(\chi) \right] - 2 L_1(\chi) \right\}, \tag{B.10c}$$

where $\chi = u/(2\mathcal{D})$.

Appendix C

Stochastic integrals and calculus

In this appendix we briefly summarize the most commonly considered stochastic integral definitions and the corresponding rules of stochastic calculus. For a more rigorous and more comprehensive introduction, we refer to Refs. [63, 64, 66, 407].

We consider a Wiener process (standard Brownian motion) $B(t)$ as defined in Section 2.1.1; i.e., the increments $\mathrm{d}B(t) := B(t + \mathrm{d}t) - B(t)$ are stochastically independent [64, 66] and characterized by the Gaussian PDF

$$\mathbb{P}\{\mathrm{d}B(t) \in [y, y + \mathrm{d}y]\} = (2\pi\,\mathrm{d}t)^{-1/2} \exp\bigl[-y^2/(2\,\mathrm{d}t)\bigr]\,\mathrm{d}y. \tag{C.1}$$

We are interested in defining integrals of the form

$$I = \int_0^t f(Y(s))\,\mathrm{d}B(s), \tag{C.2}$$

where $f(y)$ is some real-valued function and $Y(s)$ a real-valued time-dependent process. If $B(s)$ were some ordinary differentiable function of $s \in [0, t]$, then the integral in Eq. (C.2) would simply be given by[1]

$$I = \int_0^t f(Y(s))\,\dot{B}(s)\,\mathrm{d}s, \tag{C.3}$$

where $\dot{B} = \mathrm{d}B/\mathrm{d}s$. Unfortunately, $\dot{B}(s)$ is not well-defined for the Wiener process [64, 407], but it is possible to generalize the concept of integration to also include the Wiener process and other stochastic processes [64, 66, 407]. However, in contrast to the standard Riemann-Stieltjes integral (C.3), the integral with respect to a stochastic process may depend on the choice of the discretization scheme and, in particular, also require modifications of differential calculus.

[1] By writing Eq. (C.3), it is implictly assumed that $f(y)$, Y and $\dot{B}$ are sufficiently smooth functions so that this integral exists in the sense of Riemann-Stieltjes; in this case, the value of the integral (C.3) is independent of the underlying discretization scheme [66].

To illustrate these aspects for the most commonly considered stochastic integral definitions, we will always consider the following equidistant partition $\{t_0, t_1 \ldots, t_N\}$ of the time interval $[0, t]$:

$$\Delta t = t_k - t_{k-1} = t/N, \quad k = 1, \ldots, N, \qquad t_0 = 0, \qquad t_N = t. \tag{C.4}$$

C.1 Ito integral

We first summarize the properties of Ito's stochastic integral [50, 51]. Its relationship to other stochastic integrals is discussed in Section C.4.

C.1.1 One-dimensional case

The Ito stochastic integral of some real-valued function $f(Y(t))$ with respect to a standard Brownian motion process $B(t)$ over the time-interval $[0, t]$ can be defined by

$$\int_0^t f(Y(s)) * \mathrm{d}B(s) \quad := \quad \lim_{N\to\infty} \sum_{k=0}^{N-1} f(Y(t_k)) \left[B(t_{k+1}) - B(t_k)\right], \tag{C.5}$$

where the partition $\{t_0, \ldots, t_N\}$ is given by (C.4). The peculiar, defining feature of this integral is that, on the rhs. of Eq. (C.5), the argument of the function f must be evaluated at the lower boundary points t_k of the discrete intervals $[t_k, t_{k+1}]$; i.e., the definition of the Ito integral is *non-anticipating*. Accordingly, the Ito discretization scheme is also known as the *pre-point* rule.

Now consider a stochastic process $Y(t)$ which, for two given functions $A(y)$ and $C(y)$, is defined by

$$Y(t) = Y(0) + \int_0^t A(Y(s))\,\mathrm{d}s + \int_0^t C(Y(s)) * \mathrm{d}B(s), \tag{C.6}$$

and where the last term is interpreted as an Ito integral (C.5). Stochastic integral equations like Eq. (C.6) are usually abbreviated by rewriting them as an Ito *stochastic differential equation* (I-SDE)

$$\mathrm{d}Y(t) \quad = \quad A(Y)\,\mathrm{d}t + C(Y) * \mathrm{d}B(t), \tag{C.7}$$

complemented by the initial condition $Y(0)$. From the non-anticipating definition (C.5) of the Ito integral and the properties of the Wiener process it follows that [407][2]

$$\langle C(Y) * \mathrm{d}B(t) \mid Y(t) = y \rangle = 0. \tag{C.8}$$

[2] $\langle \cdot \mid Y(t) = y \rangle$ denotes the conditional expectation with respect to the Gaussian measure of the Wiener process $B(t)$.

The Fokker-Planck equation for the PDF $f(t,y)$ of the stochastic process defined by Eq. (C.7) reads

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial y}\left[-Af + \frac{1}{2}\frac{\partial}{\partial y}(C^2 f)\right], \tag{C.9}$$

where $A = A(y)$ and $C = C(y)$. The deterministic initial condition $Y(0) = y_0$ translates into $f(0,y) = \delta(y - y_0)$.

Finally, an important pecularity arises when one considers nonlinear transformations G of the stochastic process $Y(t)$. More precisley, assuming that Y is defined by the I-SDE (C.7), then the differential change of the process $Z(t) := G(Y(t))$ is given by (see, e.g., Section 4.3.2 in [407])

$$\begin{aligned} \mathrm{d}Z(t) &= G'(Y) * \mathrm{d}Y + \frac{1}{2}\, C(Y)^2\, G''(Y)\, \mathrm{d}t \\ &= \left[A(Y)\, G'(Y) + \frac{1}{2}\, C(Y)^2\, G''(Y)\right] \mathrm{d}t + C(Y)\, G'(Y) * \mathrm{d}B(t), \end{aligned} \tag{C.10}$$

where $G'(y) = \mathrm{d}G(y)/\mathrm{d}y$ and $G''(y) = \mathrm{d}^2G(y)/\mathrm{d}y^2$. Within ordinary differential calculus, the term containing G'' is absent. Equation (C.10) is usually referred to as *Ito formula*.

C.1.2 The n-dimensional case

Consider the n-dimensional stochastic process $\boldsymbol{Y}(t) = (Y^1(t), \ldots, Y^n(t))$, defined by the following n-dimensional generalization of Eq. (C.7):

$$\mathrm{d}Y^i(t) = A^i(\boldsymbol{Y})\, \mathrm{d}t + C^i{}_r(\boldsymbol{Y}) * \mathrm{d}B^r(t), \tag{C.11}$$

where $i = 1, \ldots, n$ and $r = 1, \ldots, K$. In Eq. (C.11), the Wiener processes $B^r(t)$ represent K independent noise sources, and each term $C^i{}_r(\boldsymbol{Y}) * \mathrm{d}B^r(t)$ symbolizes an Ito integral. The Fokker-Planck equation for the PDF $f(t, y^1, \ldots, y^n)$ reads

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial y^i}\left[-A^i f + \frac{1}{2}\frac{\partial}{\partial y^j}(C^i{}_r C^j{}_r f)\right]. \tag{C.12}$$

The generalized Ito-formula reads (see, e.g., Section 4.3.2 in [407])

$$\mathrm{d}G[\boldsymbol{Y}(t)] = \left[A^i\, \partial_i G + \frac{1}{2}\, C^i{}_r C^j{}_r\, \partial_i \partial_j G\right] \mathrm{d}t + C^i{}_r\, \partial_i G * \mathrm{d}B^r(t), \tag{C.13}$$

where $\partial_i := \partial/\partial y^i$.

C.2 Stratonovich-Fisk integral

Next we summarize the properties of another stochastic integral definition which was proposed by Stratonovich [57–59] and Fisk [55, 56]. In contrast to the non-anticipating Ito integral, the Stratonovich-Fisk (SF) integral is *semi-anticipating*, but satisfies the rules of ordinary stochastic calculus.

C.2.1 One-dimensional case

The SF stochastic integral of some real-valued function $f(Y(t))$ with respect to a standard Brownian (Wiener) motion process $B(t)$ over the time-interval $[0, t]$ can be defined by

$$\int_0^t f(Y(s)) \circ \mathrm{d}B(s) \;:=\; \lim_{N\to\infty} \sum_{k=0}^{N-1} \frac{1}{2} \left[f(Y(t_{k+1})) + f(Y(t_k))\right] \times \left[B(t_{k+1}) - B(t_k)\right] \tag{C.14}$$

where the partition $\{t_0, \ldots, t_N\}$ is given by (C.4). In contrast to Ito's integral (C.5), the SF definition (C.14) uses the mean of the boundary values of f on the intervals $[t_k, t_{k+1}]$; i.e., the definition of the SF integral is *semi-anticipating*. This discretization scheme is also known as the *mid-point* rule.

Similar to Eq. (C.6), we may consider a stochastic process $Y(t)$ defined by

$$Y(t) = Y(0) + \int_0^t A(Y(s))\,\mathrm{d}s + \int_0^t C(Y(s)) \circ \mathrm{d}B(s), \tag{C.15}$$

where now the last term is interpreted as an SF integral (C.14). The integral equation (C.15) can be abbreviated in terms of the equivalent SF stochastic differential equation (SF-SDE)

$$\mathrm{d}Y(t) \;=\; A(Y)\,\mathrm{d}t + C(Y) \circ \mathrm{d}B(t), \tag{C.16}$$

with initial condition $Y(0)$. From the semi-anticipating definition (C.14) of the SF integral and the properties of the Wiener process it follows that [407]

$$\langle C(Y) \circ \mathrm{d}B(t) \mid Y(t) = y \rangle = \frac{1}{2}\, C(y)\, C'(y)\,\mathrm{d}t, \tag{C.17}$$

where $C'(y) = \mathrm{d}C(y)/\mathrm{d}y$. The Fokker-Planck equation for the PDF $f(t, y)$ of the stochastic process (C.16) reads

$$\frac{\partial f}{\partial t} \;=\; \frac{\partial}{\partial y}\left[-Af + \frac{1}{2}\, C \frac{\partial}{\partial y}(Cf)\right] \tag{C.18}$$

where $A = A(y)$, $C = C(y)$ and $C' = C'(y)$. The deterministic initial condition $Y(0) = y_0$ translates into $f(0, y) = \delta(y - y_0)$.

It can be shown [66, 407] that the SF integral definition preserves the rules of ordinary stochastic calculus; i.e., if $Y(t)$ is defined by the SF-SDE (C.16), then the differential change of the process $Z(t) := G(Y(t))$ is given by (see, e.g., Section 4.3.2 in [407])

$$\begin{aligned} \mathrm{d}Z(t) &= G'(Y) \circ \mathrm{d}Y \\ &= A(Y)\, G'(Y)\, \mathrm{d}t + C(Y)\, G'(Y) \circ \mathrm{d}B(t), \end{aligned} \tag{C.19}$$

where $G'(y) = \mathrm{d}G(y)/\mathrm{d}y$.

However, as will be discussed in Section C.4, for a given SF-SDE with sufficently smooth coefficient smooth functions A and C one can always find an I-SDE, which yields the same Fokker-Planck equation. Hence, in order to describe a certain physical process, one may choose that integral definition which is most convenient for the problem under consideration.

C.2.2 The n-dimensional case

Consider the n-dimensional stochastic process $\boldsymbol{Y}(t) = (Y^1(t), \ldots, Y^n(t))$, defined by the following n-dimensional generalization of Eq. (C.16):

$$\mathrm{d}Y^i(t) = A^i(\boldsymbol{Y})\, \mathrm{d}t + C^i{}_r(\boldsymbol{Y}) \circ \mathrm{d}B^r(t), \tag{C.20}$$

where $i = 1, \ldots, n$ and $r = 1, \ldots, K$. In Eq. (C.20), the Wiener processes $B^r(t)$ represent K independent noise sources, and each term $C^i{}_r(\boldsymbol{Y}) \circ \mathrm{d}B^r(t)$ symbolizes an SF integral. The Fokker-Planck equation for the PDF $f(t, y^1, \ldots, y^n)$ reads

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial y^i}\left[-A^i f + \frac{1}{2}\, C^i{}_r \frac{\partial}{\partial y^j}(C^j{}_r f)\right], \tag{C.21}$$

and the transformation rules of ordinary differential calculus apply.

C.3 Backward Ito integral

Last but not a least, we still consider a third stochastic integral definition which is also known as the backward Ito (BI) integral [60, 64]. Its relationship to the other stochastic integrals is discussed in Section C.4.

C.3.1 One-dimensional case

The BI stochastic integral of some real-valued function $f(Y(t))$ with respect to $B(t)$ over the time-interval $[0, t]$ can be defined by

$$\int_0^t f(Y(s)) \bullet \mathrm{d}B(s) \quad := \quad \lim_{N\to\infty} \sum_{k=0}^{N-1} f(Y(t_{k+1})) \left[B(t_{k+1}) - B(t_k)\right], \tag{C.22}$$

where the partition $\{t_0, \ldots, t_N\}$ is given by (C.4). On the rhs. of Eq. (C.22), in contrast to the Ito and SF integrals, the argument of the function f must be evaluated at the upper boundary points t_{k+1} of the discrete intervals $[t_k, t_{k+1}]$; i.e., the definition of this integral is *anticipating*. This discretization scheme is also known as the *post-point* rule.

Similar to above, we may consider a stochastic process $Y(t)$ which, for two given functions $A(y)$ and $C(y)$, is defined by

$$Y(t) = Y(0) + \int_0^t A(Y(s))\,\mathrm{d}s + \int_0^t C(Y(s)) \bullet \mathrm{d}B(s), \tag{C.23}$$

and where the last term is now interpreted as a BI integral (C.22). Equation (C.6) can be abbreviated by rewriting it as a backward Ito stochastic differential equation (BI-SDE)

$$\mathrm{d}Y(t) \quad = \quad A(Y)\,\mathrm{d}t + C(Y) \bullet \mathrm{d}B(t), \tag{C.24}$$

complemented by the deterministic initial condition $Y(0)$. From the anticipating definition (C.22) of the BI integral and the properties of the Wiener process it follows that [407]

$$\langle C(Y) \bullet \mathrm{d}B(t) \mid Y(t) = y \rangle = C(y)\,C'(y)\,\mathrm{d}t. \tag{C.25}$$

The Fokker-Planck equation for the PDF $f(t, y)$ of the stochastic process defined by Eq. (C.24) reads

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial y}\left[-Af + \frac{1}{2}\,C^2 \frac{\partial}{\partial y} f\right], \tag{C.26}$$

where $A = A(y)$ and $C = C(y)$. The deterministic initial condition $Y(0) = y_0$ translates into $f(0, y) = \delta(y - y_0)$.

It can be shown that, similar to the Ito integral, also the BI integral requires a modification of differential calculus. More precisley, assuming that Y is defined by the BI-SDE (C.24), the differential change of the process $Z(t) := G(Y(t))$ is given by

$$\begin{aligned} \mathrm{d}Z(t) &= G'(Y) \bullet \mathrm{d}Y - \frac{1}{2}\, C(Y)^2\, G''(Y)\, \mathrm{d}t \\ &= \left[A(Y)\, G'(Y) - \frac{1}{2}\, C(Y)^2\, G''(Y)\right] \mathrm{d}t + C(Y)\, G'(Y) \bullet \mathrm{d}B(t), \end{aligned} \tag{C.27}$$

where $G'(y) = \mathrm{d}G(y)/\mathrm{d}y$ and $G''(y) = \mathrm{d}^2G(y)/\mathrm{d}y^2$.

C.3.2 The n-dimensional case

Consider the n-dimensional stochastic process $\boldsymbol{Y}(t) = (Y^1(t), \ldots, Y^n(t))$, defined by the following n-dimensional generalization of Eq. (C.24):

$$\mathrm{d}Y^i(t) = A^i(\boldsymbol{Y})\, \mathrm{d}t + C^i{}_r(\boldsymbol{Y}) \bullet \mathrm{d}B^r(t), \tag{C.28}$$

where $i = 1, \ldots, n$ and $r = 1, \ldots, K$. In Eq. (C.28), the Wiener processes $B^r(t)$ represent K independent noise sources, and each term $C^i{}_r(\boldsymbol{Y}) \bullet \mathrm{d}B^r(t)$ symbolizes a BI integral. The Fokker-Planck equation for the associated PDF $f(t, y^1, \ldots, y^n)$ reads

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial y^i}\left[-A^i f + \frac{1}{2}\, C^i{}_r C^j{}_r \frac{\partial}{\partial y^j} f\right]. \tag{C.29}$$

The generalized backward Ito-formula reads

$$\mathrm{d}G[\boldsymbol{Y}(t)] = \left[A^i\, \partial_i G - \frac{1}{2}\, C^i{}_r C^j{}_r\, \partial_i \partial_j G\right] \mathrm{d}t + C^i{}_r\, \partial_i G \bullet \mathrm{d}B^r(t), \tag{C.30}$$

where $\partial_i := \partial/\partial y^i$.

C.4 Comparison of stochastic integrals

As anticipated in the preceding sections, the three different stochastic integrals/SDEs may be transformed into each other. In particular, a given Fokker-Planck equation can usually be realized by any of three SDE types, upon choosing the coefficient functions appropriately. To illustrate this by example, we reconsider the n-dimensional SDEs from above, assuming identical noise coefficients $C^i{}_r$ but different drift coefficients $A^i_{*|\circ|\bullet}(\boldsymbol{Y})$, respectively, i.e.

$$\mathrm{d}Y^i(t) = A^i_*(\boldsymbol{Y})\, \mathrm{d}t + C^i{}_r(\boldsymbol{Y}) * \mathrm{d}B^r(t), \tag{C.31a}$$

$$\mathrm{d}Y^i(t) = A^i_\circ(\boldsymbol{Y})\, \mathrm{d}t + C^i{}_r(\boldsymbol{Y}) \circ \mathrm{d}B^r(t), \tag{C.31b}$$

$$\mathrm{d}Y^i(t) = A^i_\bullet(\boldsymbol{Y})\, \mathrm{d}t + C^i{}_r(\boldsymbol{Y}) \bullet \mathrm{d}B^r(t), \tag{C.31c}$$

where $i = 1, \ldots, n$ and $r = 1, \ldots, K$. We would like to determine the drift coefficients such that these three different types of SDEs describe the same n-dimensional stochastic process $\boldsymbol{Y}(t) = (Y^1(t), \ldots, Y^n(t))$ on the level of the Fokker-Planck equation.[3] We distinguish three cases.

Equation (C.31a) **is given:** The Fokker-Planck equation for the PDF $f(t, y^1, \ldots, y^n)$ of the Ito process from Eq. (C.31a) reads

$$\partial_t f = \partial_i \left[-A^i_* f + \frac{1}{2} \partial_j (C^i{}_r C^j{}_r f) \right], \tag{C.32}$$

where $\partial_t = \partial/\partial t$ and $\partial_i := \partial/\partial y^i$. Next we rewrite the SF Fokker-Planck equation (C.21), corresponding to Eq. (C.31b), in the equivalent form

$$\partial_t f \;=\; \partial_i \left[-\left(A^i_\circ + \frac{1}{2} C^j{}_r \partial_j C^i{}_r \right) f + \frac{1}{2} \partial_j (C^i{}_r C^j{}_r f) \right]. \tag{C.33}$$

Similarly, the BI Fokker-Planck equation (C.29), corresponding to Eq. (C.31c), can be rewritten as

$$\partial_t f \;=\; \partial_i \left\{ -\left[A^i_\bullet + \frac{1}{2} \partial_j (C^i{}_r C^j{}_r) \right] f + \frac{1}{2} \partial_j (C^i{}_r C^j{}_r f) \right\}. \tag{C.34}$$

Upon comparing Eqs. (C.33) and (C.34) with Eq. (C.32), we see that Eqs. (C.31b) and (C.31c) describe the same process like Eq. (C.31a), if we fix

$$A^i_\circ = A^i_* - \frac{1}{2} C^j{}_r \partial_j C^i{}_r \,, \qquad\qquad A^i_\bullet = A^i_* - \frac{1}{2} \partial_j (C^i{}_r C^j{}_r). \tag{C.35}$$

Equation (C.31b) **is given:** The Fokker-Planck equation for the PDF $f(t, y^1, \ldots, y^n)$ of the SF process from Eq. (C.31b) reads

$$\partial_t f \;=\; \partial_i \left[-A^i_\circ f + \frac{1}{2} C^i{}_r \partial_j (C^j{}_r f) \right], \tag{C.36}$$

Next we rewrite the Ito Fokker-Planck equation (C.12), corresponding to Eq. (C.31a), as

$$\partial_t f \;=\; \partial_i \left[-\left(A^i_* - \frac{1}{2} C^j{}_r \partial_j C^i{}_r \right) f + \frac{1}{2} C^i{}_r \partial_j (C^j{}_r f) \right], \tag{C.37}$$

Similarly, the BI Fokker-Planck equation (C.29), corresponding to Eq. (C.31c), can also be written as

$$\partial_t f \;=\; \partial_i \left\{ -\left[A^i_\bullet + \frac{1}{2} C^i{}_r \partial_j C^j{}_r \right] f + \frac{1}{2} C^i{}_r \partial_j (C^j{}_r f) \right\}. \tag{C.38}$$

[3]For most practical purposes, two Markovian stochastic processes can be considered as physically equivalent if their PDFs are governed by the same Fokker-Planck equation.

Thus, upon comparing Eqs. (C.37) and (C.38) with Eq. (C.36), we see that Eqs. (C.31a) and (C.31c) describe the same process like Eq. (C.31b), if we fix

$$A^i_* = A^i_\circ + \frac{1}{2}\, C^j{}_r \partial_j C^i{}_r \,, \qquad\qquad A^i_\bullet = A^i_\circ - \frac{1}{2}\, C^i{}_r \partial_j C^j{}_r. \tag{C.39}$$

Equation (C.31c) **is given:** The Fokker-Planck equation for the PDF $f(t, y^1, \ldots, y^n)$ of the BI process from Eq. (C.31c) reads

$$\partial_t f = \partial_i \left[-A^i_\bullet f + \frac{1}{2}\, C^i{}_r C^j{}_r \partial_j f \right]. \tag{C.40}$$

Next we rewrite the Ito Fokker-Planck equation (C.12), corresponding to Eq. (C.31a), as

$$\partial_t f \;\; = \;\; \partial_i \left\{ - \left[A^i_* - \frac{1}{2}\, \partial_j (C^i{}_r C^j{}_r) \right] f + \frac{1}{2}\, C^i{}_r C^j{}_r \partial_j f \right\}, \tag{C.41}$$

Similarly, the SF Fokker-Planck equation (C.21), corresponding to Eq. (C.31c), can also be written as

$$\partial_t f \;\; = \;\; \partial_i \left[- \left(A^i_\circ - \frac{1}{2}\, C^i{}_r \partial_j C^j{}_r \right) f + \frac{1}{2}\, C^i{}_r C^j{}_r \partial_j f \right]. \tag{C.42}$$

Thus, upon comparing Eqs. (C.41) and (C.42) with Eq. (C.40), we see that Eqs. (C.31a) and (C.31b) describe the same process like Eq. (C.31c), if we fix

$$A^i_* = A^i_\bullet + \frac{1}{2}\, \partial_j (C^i{}_r C^j{}_r) \,, \qquad\qquad A^i_\circ = A^i_\bullet + \frac{1}{2}\, C^i{}_r \partial_j C^j{}_r. \tag{C.43}$$

To summarize, by means of Eqs. (C.35), (C.39) and (C.43) one can change between the different forms of stochastic integration and stochastic differential calculus, respectively. Each SDE type has advantages and disadvantages: The Ito formalism is well suited for numerical simulations [63, 79, 336] and yields a vanishing noise contribution to conditional expectations of the form (C.8). The Stratonovich-Fisk approach is more difficult to implement numerically, but preserves the rules of ordinary differential calculus (in contrast to Ito/backward Ito integration). Finally, within the backward Ito scheme, fluctuation dissipation relations take a particularly elegant form (cf. Sec. 6.2 in Ref. [67], and Ref. [101]).

C.5 Numerical integration

A detailed introduction to the numerical simulation of SDEs can be found in [63, 79, 336]. A simple Monte-Carlo algorithm for numerically integrating Eqs. (C.31) follows directly from the definition of the stochastic integrals. The corresponding discretization scheme, which works sufficiently well for many purposes, reads

$$
\begin{aligned}
Y^i(t+\Delta t) - Y^i(t) &= A^i_*(\boldsymbol{Y}(t))\,\Delta t + C^i{}_r(\boldsymbol{Y}(t))\,\Delta B^r(t), && \text{(C.44a)}\\
Y^i(t+\Delta t) - Y^i(t) &= A^i_\circ(\boldsymbol{Y}(t))\,\Delta t + \\
&\qquad \frac{1}{2}[C^i{}_r(\boldsymbol{Y}(t+\Delta t)) + C^i{}_r(\boldsymbol{Y}(t))]\,\Delta B^r(t), && \text{(C.44b)}\\
Y^i(t+\Delta t) - Y^i(t) &= A^i_\bullet(\boldsymbol{Y}(t))\,\Delta t + C^i{}_r(\boldsymbol{Y}(t+\Delta t))\,\Delta B^r(t). && \text{(C.44c)}
\end{aligned}
$$

Here, the $\Delta B^r(t)$ are random numbers, sampled from a Gaussian normal distribution with density

$$
\mathbb{P}[\Delta B^r(t)] = \left(\frac{1}{2\pi\Delta t}\right)^{1/2} \exp\left\{-\frac{[\Delta B^r(t)]^2}{2\Delta t}\right\}. \tag{C.45}
$$

As evident from Eqs. (C.44), for given functions A^i_* and $C^i{}_r$, the discretized Ito SDE (C.44a) allows for calculating the values $Y^i(t+\Delta t)$ directly from the preceding values $Y^i(t)$. By contrast, the discretized SF SDEs (C.44b) and BI SDEs (C.44c) are implicit equations, which must be solved for $Y^i(t+\Delta t)$. The latter difficulty can be avoided by transforming a given SF/BI SDE to the corresponding Ito SDE by means of Eqs. (C.35), (C.39) and (C.43).

Appendix D

Relativistic Brownian motion processes in higher space dimensions

Section D.1 summarizes the Langevin equations of d-dimensional relativistic Brownian motion models in the lab frame. Moving observers will be briefly discussed in Section D.2.

D.1 Lab frame

In the lab frame Σ, the d-dimensional stochastic motion of the relativistic Brownian particle can be described by the Langevin equations

$$\mathrm{d}X^i(t) = (P^i/P^0)\,\mathrm{d}t, \tag{D.1a}$$

$$\mathrm{d}P^i(t) = -a^i{}_j P^j \mathrm{d}t + c^i{}_r \bullet \mathrm{d}B^r(t). \tag{D.1b}$$

where $P^0 = (M^2 + \boldsymbol{P}^2)^{1/2}$, $i = 1, \ldots, d$, and $r = 1, \ldots, d$. Here, for simplicity, we assumed that one independent noise source couples to each momentum component; more general models may feature multiple noise sources. Equations (D.1) constitute a special case of the general post-point SDE (C.28), upon identifying $n = 2d$, $\boldsymbol{Y} = (X^1, \ldots, X^d, P^1, \ldots, P^d)$ and

$$A^i = P^i/P^0, \qquad C^i{}_r = 0, \qquad A^{i+d} = -a^i{}_j P^j, \qquad C^{i+d}{}_r = c^i{}_r \tag{D.2}$$

for $i = 1, \ldots, d$. From Eq. (C.29), the Fokker-Planck equation corresponding to the stochastic process (D.1) is obtained as

$$\left[\frac{\partial}{\partial t} + \frac{p^i}{p^0}\frac{\partial}{\partial x^i}\right] f = \frac{\partial}{\partial p^i}\left[a^i{}_j p^j f + \frac{1}{2} c^i{}_r c^k{}_r \frac{\partial}{\partial p^k} f\right] \tag{D.3}$$

with $f(t, \boldsymbol{x}, \boldsymbol{p})$ denoting the phase space PDF of the relativistic Brownian particle in Σ, and $p^0 = (M^2 + \boldsymbol{p}^2)^{1/2}$ the relativistic energy.

In general, the friction and noise coefficients, $a^i{}_j$ and $c^i{}_j$ may be functions of the momentum and position coordinates, $\boldsymbol{P} = (P^i)$ and $\boldsymbol{X} = (X^i)$, and of the lab time t. Restricting the discussion to a heat bath that is stationary, isotropic and homogeneous in the lab frame Σ, the coefficient matrices take the simplified diagonal form

$$a^i{}_j = \alpha\,\delta^i{}_j\ , \qquad\qquad c^i{}_j = (2D)^{1/2}\,\delta^i{}_j, \tag{D.4a}$$

where the functions α and D depend only on the Brownian particles' *absolute* momentum – or, equivalently, on its relativistic energy $P^0 = (M^2 + \boldsymbol{P}^2)^{1/2}$. Denoting by $U^\beta = (1, \boldsymbol{0})$ the mean four velocity vector of the heat bath in Σ, we have $P^0 = -U^\beta P_\beta$ and may therefore write

$$\alpha = \alpha(U^\lambda P_\lambda)\ , \qquad\qquad D = D(U^\lambda P_\lambda). \tag{D.4b}$$

For an isotropic background modelled by Eqs. (D.4), the Langevin equations (D.1) can be rewritten as

$$\begin{aligned} \mathrm{d}X^i(t) &= (P^i/P^0)\,\mathrm{d}t, & \text{(D.5a)}\\ \mathrm{d}P^i(t) &= -\alpha P^i \mathrm{d}t + (2D)^{1/2} \bullet \mathrm{d}B^i(t). & \text{(D.5b)} \end{aligned}$$

and the Fokker-Planck equation (D.3) simplifies to

$$\left[\frac{\partial}{\partial t} + \frac{p^i}{p^0}\frac{\partial}{\partial x^i}\right] f \;=\; \frac{\partial}{\partial p^i}\left[\alpha p^i f + D\frac{\partial}{\partial p^i} f\right], \tag{D.6}$$

where $\alpha = \alpha(U^\lambda p_\lambda)$ and $D = D(U^\lambda p_\lambda)$. In particular, if α and D satisfy the generalized fluctuation-dissipation relation

$$\alpha p^0 = D\beta, \tag{D.7}$$

and if the motion is restricted to a finite volume $\mathcal{V}$, then the stationary solution $f_\infty(\boldsymbol{x}, \boldsymbol{p})$ of Eq. (D.6) is given by a spatially homogeneous Jüttner distribution [167]

$$f_\infty(\boldsymbol{x}, \boldsymbol{p}) = \mathcal{N}\ \exp[-\beta(M^2 + \boldsymbol{p}^2)^{1/2}]\ I_{\mathcal{V}}(\boldsymbol{x}), \tag{D.8}$$

where $I_{\mathcal{V}}(\boldsymbol{x}) := 1$ if $\boldsymbol{x} \in \mathcal{V}$, and $I_{\mathcal{V}}(\boldsymbol{x}) := 0$ otherwise.

D.2 Moving observer

The equations in Section D.1 refer to the lab frame Σ, defined as the rest frame of the heat bath. An observer at rest in this lab frame would measure the PDF f governed by the Fokker-Planck equations (D.3) and (D.6). What is the corresponding PDF f' seen by a moving observer? According to van Kampen [138], the one-particle phase space density transforms as a Lorentz scalar; i.e.,

$$f'(t', \boldsymbol{x}', \boldsymbol{p}') = f(t(t', \boldsymbol{x}'), \boldsymbol{x}(t', \boldsymbol{x}'), \boldsymbol{p}(\boldsymbol{p}')), \tag{D.9a}$$

and, conversely,

$$f(t, \boldsymbol{x}, \boldsymbol{p}) = f'(t'(t, \boldsymbol{x}), \boldsymbol{x}'(t, \boldsymbol{x}), \boldsymbol{p}'(\boldsymbol{p})), \tag{D.9b}$$

where $(t', \boldsymbol{x}', \boldsymbol{p}')$ and $(t, \boldsymbol{x}, \boldsymbol{p})$ are related by the Lorentz transformation

$$x'^{\lambda}(t, \boldsymbol{x}) = \Lambda^{\lambda}{}_{0} t + \Lambda^{\lambda}{}_{i} x^{i}, \tag{D.10a}$$

$$p'^{i}(\boldsymbol{p}) = \Lambda^{i}{}_{0} (m^2 + \boldsymbol{p}^2)^{1/2} + \Lambda^{i}{}_{j} p^{j}. \tag{D.10b}$$

Specifically, for an observer moving at constant velocity $\boldsymbol{w}$ through the lab frame, the corresponding Lorentz boost matrix elements read [cf. Eq. (A.15)]

$$\Lambda^{0}{}_{0} = \gamma, \qquad \Lambda^{i}{}_{0} = \Lambda^{0}{}_{i} = -\gamma w^{i}, \qquad \Lambda^{i}{}_{j} = \delta^{i}{}_{j} + w^{i} w^{j} \frac{(\gamma - 1)}{\boldsymbol{w}^2}. \tag{D.11}$$

Hence, in order to find f', it suffices to solve the the Fokker-Planck equations (D.3) or (D.6) in lab frame Σ, e.g., for a given t-simultaneous initial condition $f(0, \boldsymbol{x}, \boldsymbol{p})$, and to insert the solution into (D.9a).

Bibliography

[1] A. Einstein. Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt. *Ann. Phys. (Leipzig)*, 17(4):132–148, 1905.

[2] A. Einstein. Zur Elektrodynamik bewegter Körper. *Ann. Phys. (Leipzig)*, 17(4):891–921, 1905.

[3] A. Einstein. Ist die Trägheit eines Körpers von seinem Energiegehalt abhängig? *Ann. Phys. (Leipzig)*, 18(4):639–641, 1905.

[4] A. Einstein. Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspensierten Teilchen. *Ann. Phys. (Leipzig)*, 17(4):549–560, 1905.

[5] C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation.* W. H. Freeman and Co., New York, 2000. 23rd printing.

[6] S. Weinberg. *Gravitation and Cosmology.* John Wiley & Sons, 1972.

[7] C. Lämmerzahl. Special Relativity and Lorentz Invariance. *Ann. Phys. (Leipzig)*, 14(1-3):71–102, 2005.

[8] R. U. Sexl and H. K. Urbantke. *Relativity, Groups, Particles.* Springer Physics. Springer, Wien, 2001.

[9] S. Weinberg. *The Quantum Theory of Fields*, volume 1. Cambridge University Press, Cambridge, 2002.

[10] J. Dunkel and P. Hänggi. Theory of the relativistic Brownian motion: The (1+1)-dimensional case. *Phys. Rev. E*, 71:016124, 2005.

[11] J. Dunkel and P. Hänggi. Theory of the relativistic Brownian motion: The (1+3)-dimensional case. *Phys. Rev. E*, 72:036106, 2005.

[12] J. Dunkel, S. Hilbert, and P. Hänggi. Langevin-Gleichungen mit nichtlinearer Reibung. In T. Pöschel, H. Malchow, and L. Schimansky-Geier, editors, *Irreversible Prozesse und Selbstorganisation*, pages 11–21. Logos-Verlag, Berlin, 2006. ISBN 3-8325-1350-7.

[13] J. Dunkel and P. Hänggi. Relativistic Brownian motion: From a microscopic binary collision model to the Langevin equation. *Phys. Rev. E*, 74:051106, 2006. *Erratum*, Phys. Rev. E 74:069902(E), 2006.

[14] J. Dunkel and P. Hänggi. One-dimensional nonrelativistic and relativistic Brownian motions: A microscopic collision model. *Physica A*, 374(2):559–572, 2007.

[15] J. Dunkel, P. Talkner, and P. Hänggi. Relativistic diffusion processes and random walk models. *Phys. Rev. D*, 75:043001, 2007.

[16] J. Dunkel, P. Talkner, and P. Hänggi. Relative entropy, Haar measures and relativistic canonical velocity distributions. *New J. Phys.*, 9:144, 2007.

[17] D. Cubero, J. Casado-Pascual, J. Dunkel, P. Talkner, and P. Hänggi. Thermal equilibrium and statistical thermometers in special relativity. *Phys. Rev. Lett.*, 99:170601, 2007.

[18] F. Debbasch, K. Mallick, and J. P. Rivet. Relativistic Ornstein-Uhlenbeck Process. *J. Stat. Phys*, 88:945–966, 1997.

[19] F. Debbasch and J. P. Rivet. A diffusion equation from the relativistic Ornstein-Uhlenbeck process. *J. Stat. Phys*, 90:1179–1199, 1998.

[20] F. Debbasch. Relativistic stochastic processes: A review. *AIP Conf. Proc.*, 861:488–493, 2006.

[21] C. Barbachoux, F. Debbasch, and J. P. Rivet. The spatially one-dimensional relativistic Ornstein-Uhlenbeck process in an arbitrary inertial frame. *Eur. Phys. J. B*, 19:37–47, 2001.

[22] C. Barbachoux, F. Debbasch, and J. P. Rivet. Covariant kolmogorov equation and entropy current for the relativistic Ornstein-Uhlenbeck process. *Eur. Phys. J. B*, 20:487–496, 2001.

[23] J. Angst and J. Franchi. Central limit theorem for a class of relativistic diffusions. *J. Math. Phys.*, 48(8):083101, 2007.

[24] B. Lindner. Exact solution for the diffusion coefficient of nonlinear brownian motion. *New J. Phys.*, 9:136, 2007.

[25] J. Ingen-Housz. Bemerkungen über den Gebrauch des Vergrösserungsglases. In N. C. Molitor, editor, *Vermischte Schriften physisch-medicinischen Inhalts*. Christian Friderich Wappler, Wien, 1784.

[26] J. Ingen-Housz. *Nouvelles expériences et observations sur divers objets de physique*, volume 2. Théophile Barrois le jeune, Paris, 1789. pages 1-5.

[27] R. Brown. A brief account of microscopical observations made in the months of June, July and August, 1827, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies. *Philos. Mag*, 4:161–173, 1828. reprinted in Edinburgh New Philos. J. 5:358-371, 1928.

[28] W. Sutherland. A dynamical theory of diffusion for non-electrolytes and the molecular mass of albumin. *Philos. Mag*, 9:781–785, 1905.

[29] A. Einstein and M. von Smoluchowski. *Untersuchungen über die Theorie der Brownschen Bewegung/Abhandlungen über die Brownsche Bewegung und verwandte Erscheinungen*, volume 199. Harri Deutsch, Frankfurt, 3 edition, 1999.

[30] J. Perrin. Mouvement brownien et réalité moléculaire. *Ann. Chim. Phys.*, 18:5–114, 1909.

[31] P. Langevin. Sur la théorie du mouvement brownien. *Comptes rendus Acad. Sci. (Paris)*, 146:530–533, 1908.

[32] D. S. Lemons and A. Gythiel. Paul Langevins's 1908 paper "On the Theory of Brownian Motion". *Am J. Phys.*, 65(11):1079–1081, 1997.

[33] A. Fokker. Die mittlere Energie rotierender elektrischer Dipole im Strahlungsfeld. *Ann. Phys.*, 43(4):810–820, 1914.

[34] M. Planck. Über einen Satz der statistischen Dynamik und seine Erweiterung in der Quantentheorie. *Sitzungsber. Preuss. Akad. Wiss. Berlin*, 1917.

[35] O. Klein. *Zur statistischen Theorie der Suspensionen und Lösungen.* PhD thesis, Stockholm Högskola, 1921. published in Ark. Mat. Astron. Fys. 16(5):1-51, 1922. Almqvist & Wiksell, Uppsala.

[36] G. E. Uhlenbeck and L. S. Ornstein. On the Theory of the Brownian Motion. *Phys. Rev.*, 36:823–841, 1930.

[37] H. A. Kramers. Brownian motion in a field of force and the diffusion model of chemical reactions. *Physica*, 7(4):284–304, 1940.

[38] S. Chandrasekhar. Stochastic Problems in Physics and Astronomy. *Rev. Mod. Phys.*, 15(1):1–89, 1943.

[39] M. C. Wang and G. E. Uhlenbeck. On the Theory of the Brownian Motion II. *Rev. Mod. Phys.*, 17:323–342, 1945.

[40] L. Bachelier. Théorie de la spéculation. *Annales Scientifiques de l'É. N. S.*, 3(17):21–86, 1900.

[41] N. Wiener. Differential space. *J. Math. Phys.*, 2:131–174, 1923.

[42] N. Wiener. Generalized harmonic analysis. *Acta Math.*, 55:117–258–, 1930.

[43] R. E. A. C. Paley, N. Wiener, and A. Zygmund. Notes on random functions. *Ann. Math.*, 37(1):647–668, 1933.

[44] A. Kolmogoroff. Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung. *Math. Annalen*, 104(4):415–458, 1931.

[45] A. Kolmogoroff. Zur Theorie der stetigen zufälligen Prozesse. *Math. Annalen*, 108(1):149–160, 1933.

[46] A. Kolmogoroff. Zur Umkehrbarkeit der statistischen Naturgesetze. *Math. Annalen*, 113(1):766–772, 1937.

[47] W. Feller. Zur Theorie der stochastischen Prozesse (Existenz- und Eindeutigkeitssätze). *Ann. Math.*, 113(1):113–160, 1936.

[48] P. Lévy. Sur certains processus stochastique homogènes. *Compositio Math.*, 7:283–339, 1939.

[49] P. Lévy. *Processus Stochastique et Mouvement Brownien*. Gauthier-Villars, Paris, 1948.

[50] K. Ito. Stochastic integral. *Proc. Imp. Acad. Tokyo*, 20:519–524, 1944.

[51] K. Ito. On stochastic differential equations. *Mem. Amer. Mathem. Soc.*, 4:51–89, 1951.

[52] I. I. Gihman. On a method of constructing random processes. *Dokl. Akad. Nauk. SSSR*, 58:961–964, 1947Gi. In Russian.

[53] I. I. Gihman. Certain differential equations with random functions. *Ukrain. Math. Zh.*, 2(3):45–69, 1950. In Russian.

[54] I. I. Gihman. On the theory of differential equations of random processes. *Ukrain. Math. Zh.*, 2(4):37–63, 1950. In Russian.

[55] D. Fisk. *Quasi-martingales and stochastic integrals.* PhD thesis, Michigan State University, Dept. of Statistics, 1963.

[56] D. Fisk. Quasimartingales. *Trans. Amer. Math. Soc.*, 120:369–389, 1965.

[57] R. L. Stratonovich. A New Representation for Stochastic Integrals and Equations. *Vestnik Moskov. Univ., Ser. I: Mat., Mekh.*, 1:3–12, 1964.

[58] R. L. Stratonovich. A New Representation for Stochastic Integrals and Equations. *SIAM J. Control*, 4:362–371, 1966.

[59] R. L. Stratonovich. *Conditional Markov processes and their application to the theory of optimal control.* American Elsevier Pub. Co., 1968.

[60] H. P. MacKean Jr. *Stochastic Integrals.* Academic Press, New York, 1969.

[61] E. Nelson. *Theories of Brownian Motion.* Princeton University Press, Princeton, 1967.

[62] D. R. Cox and H. D. Miller. *The Theory of Stochastic Processes.* Chapman and Hall, Oxford, 1972.

[63] H. Risken and T. Frank. *The Fokker-Planck Equation: Methods of Solutions and Applications.* Springer Series in Synergetics. Springer, Berlin, 1996.

[64] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus.* Number 113 in Graduate Texts in Mathematics. Springer, New York, Berlin, 2 edition, 1991.

[65] N. G. Van Kampen. *Stochastic Processes in Physics and Chemistry.* North-Holland Personal Library, Amsterdam, 2003.

[66] M. Grigoriu. *Stochastic Calculus: Applications in Science and Engineering.* Birkhäuser, Boston, Basel, Berlin, 2002.

[67] P. Hänggi and H. Thomas. Stochastic processes: Time evolution, symmetries and linear response. *Phys. Rep.*, 88(4):207–319, 1982.

[68] P. Hänggi, P. Talkner, and M. Borkovec. Reaction rate theory: fifty years after Kramers. *Rev. Mod. Phys.*, 62(2):251–341, 1990.

[69] P. Reimann. Brownian motors: noisy transport far from equilibrium. *Phys. Rep.*, 361(2):57–265, 2002.

[70] R. D. Astumian und P. Hänggi. Brownian Motors. *Physics Today*, 55(11):33–39, 2002.

[71] P. Hänggi and F. Marchesoni. 100 Years of Brownian Motion. *Chaos*, 15:026101, 2005.

[72] E. Frey and K. Kroy. Brownian motion: a paradigm of soft matter and biological physics. *Ann. Phys. (Leipzig)*, 14(1-3):20–50, 2005.

[73] I. M. Sokolov and J. Klafter. From diffusion to anomalous diffusion: A century after Einstein's Brownian motion. *Chaos*, 15:026103, 2005.

[74] W. Ebeling and I. M. Sokolov. *Statistical thermodynamics and stochastic theory of nonequilibrium systems*. Series on Advances in Statistical Mechanics. World Scientific Publishing, Singapore, 2005.

[75] L. J. S. Allen. *Stochastic Processes with Applications to Biology*. Prentice-Hall, Upper Saddle River, NJ, 2002.

[76] D. Wilkinson. *Stochastic Modelling for Systems Biology*, volume 11 of *Mathematical and Computational Biology Series*. Chapman & Hall/CRC, London, 2006.

[77] R. J. Elliott and P. E. Kopp. *Mathematics of Financial Markets*. Springer-Verlag, New York, 1999.

[78] J.-P. Bouchaud and M. Potters. *Theory of Financial Risks: From Statistical Physics to Risk Management*. Cambridge University Press, Cambridge, 2001.

[79] P. Glasserman. *Monte Carlo Methods in Financial Engineering*. Number 53 in Applications of Mathematics. Springer, New York, 2004.

[80] N. N. Bogolyubov. An elementary example of establishing statistical equilibrium in a system coupled to a thermostat. *Publ. Acad. Sci. Ukr. SSR Kiev*, pages 115–137, 1945. In Russian.

[81] V. B. Magalinskii. Dynamical model in the theory of the Brownian motion. *Sov. Phys. JETP*, 9:1381–1382, 1959.

[82] G. W. Ford, M. Kac, and P. Mazur. Statistical Mechanics of Assemblies of Coupled Oscillators. *J. Math. Phys.*, 6(4):505–515, 1965.

[83] E. Cortes, B. J. West, and K. Lindenberg. On the generalized Langevin equation: Classical and quantum mechanical. *J. Chem. Phys.*, 82(6):2708–2717, 1985.

[84] E. Pollak. Theory of activated rate processes: A new derivation of Kramers' expression. *J. Chem. Phys.*, 85:865–867, 1986.

[85] A. M. Levine, M. Shapiro, and E. Pollak. Hamiltonian theory for vibrational dephasing rates of small molecules in liquids. *J. Chem. Phys.*, 88(3):1959–1966, 1988.

[86] P. Hänggi. Generalized Langevin equations: A useful tool for the perplexed modeller of non-equilibrium fluctuations. In *Stochastic dynamics*, volume 484 of *Lecture Notes in Physics*, pages 15–22. Springer, Heidelberg, 1997.

[87] I. R. Senitzky. Dissipation in Quantum Mechanics. The Harmonic Oscillator. *Phys. Rev.*, 119(2):670–679, 1960.

[88] I. R. Senitzky. Dissipation in Quantum Mechanics. The Harmonic Oscillator. II. *Phys. Rev.*, 124(3):642–648, 1961.

[89] A. O. Caldeira and A. J. Leggett. Path Integral Approach to Quantum Brownian Motion. *Physica*, 121A:587–616, 1983.

[90] P. Talkner. The failure of the quantum regression hypothesis. *Ann. Phys. (N.Y.)*, 167(2):390–436, 1986.

[91] H. Grabert, U. Weiss, and P. Talkner. Quantum theory of the damped harmonic oscillator. *Z. Phys. B Cond. Mat.*, 55:87–94, 1984.

[92] A. O. Caldeira and A. J. Leggett. Influence of damping on quantum interference: An exactly soluble model. *Phys. Rev. A*, 31(2):1059–1066, 1983.

[93] G. W. Ford and M. Kac. On the Quantum Langevin Equation. *J. Stat. Phys.*, 46(5/6):803–810, 1987.

[94] G. W. Ford, J. T. Lewis, and R. F. O' Connell. Quantum Langevin equation. *Phys. Rev. A*, 37(11):4419–4428, 1988.

[95] P. Pechukas. Quantum Brownian Motion. In W. Gans, A. Blumen, and A. Amann, editors, *Large-Scale Molecular Systems*, NATO ASI B258, page 123, New York, 1991. Plenum.

[96] S. Tsonchev and P. Pechukas. Binary collision model for quantum Brownian motion. *Phys. Rev. E*, 61:6171–6182, 2000.

[97] S. Kohler, J. Lehmann, and P. Hänggi. Driven quantum transport on the nanoscale. *Phys. Rep.*, 406:379–443, 2005.

[98] P. Hänggi and G. L. Ingold. Fundamental aspects of quantum Brownian motion. *Chaos*, 15:026105, 2005.

[99] P. Hänggi and G. L. Ingold. Quantum Brownian motion and the third law of thermodynamics. *Acta Phys. Polonica B*, 37(5):1537–1549, 2006.

[100] R. F. Fox. Gaussian stochastic processes in physics. *Phys. Rep.*, 48(3):179–283, 1978.

[101] Yu. L. Klimontovich. Nonlinear Brownian Motion. *Physics-Uspekhi*, 37(8):737–766, 1994.

[102] Y. L. Klimontovich. *Statistical Theory of Open Systems*. Kluwer Acad. Publ., Dordrecht, 1995.

[103] P. Hänggi and P. Jung. Colored Noise in Dynamical Systems. *Adv. Chem. Phys.*, 89:239–326, 1995.

[104] Y. Abe, S. Ayik, P. G. Reinhard, and E. Suraud. On stochastic approaches of nuclear dynamics. *Phys. Rep.*, 275(2-3):49–196, 1996.

[105] J.-P. Bouchaud and A. Georges. Anomalous diffusion in disordered media: Statistical mechanisms, models and physical applications. *Phys. Rep.*, 195(4-5):127–293, 1990.

[106] R. Metzler and J. Klafter. The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys. Rep.*, 339(1):1–77, 2000.

[107] H. Grabert, P. Schramm, and G.-L. Ingold. Quantum brownian motion: The functional integral approach. *Phys. Rep.*, 168(3):115–207, 1988.

[108] M. Grifoni and P. Hänggi. Driven quantum tunneling. *Phys. Rep.*, 304:229–354, 1998.

[109] A. Einstein. Über das Relativitätsprinzip und die aus demselben gezogenen Folgerungen. *Jahrbuch der Radioaktivität und Elektronik*, 4:411–462, 1907.

[110] M. Planck. Zur Dynamik bewegter Systeme. *Sitzungsber. Preuss. Akad. Wiss. Berlin*, pages 542–570, 1907.

[111] M. Planck. Zur Dynamik bewegter Systeme. *Ann. Phys. (Leipzig)*, 26:1–34, 1908.

[112] W. Pauli. Relativitätstheorie. In *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*, volume V/2, pages 539–775. Teubner, Leipzig, 1921.

[113] R. C. Tolman. *Relativity, Thermodynamics, and Cosmology.* Clarendon Press, Oxford, 1934.

[114] D. Van Dantzig. On the phenomenological thermodynamics of moving matter. *Physica*, 6:673–704, 1939.

[115] H. Ott. Lorentz-Transformation der Wärme und der Temperatur. *Z. Phys.*, 175:70–104, 1963.

[116] A. Børs. Note on the temperature of a gas moving at relativistic speed. *Proc. Roy. Soc. (London)*, 86:1141–1143, 1965.

[117] A. Gamba. Relativistic transformations of thermodynamical quantities. *Nuovo Cimento*, 37(4):1792–1794, 1965.

[118] W. G. Sutcliffe. Lorentz transformations of thermodynamic quantities. *Nuovo Cimento B*, 9:683–686, 1965.

[119] H. Arzelies. Transformation relativiste de la temperature et de quelques autres grandeurs thermodynamiques. *Nuovo Cimento*, 35:792–804, 1965.

[120] H. Arzelies. Comment on dr. kibble's article. *Nuovo Cimento B*, 41(1):81–82, 1966.

[121] F. Rohrlich. True and Apparent Transformations, Classical Electrons and Relativistic Thermodynamics. *Nuovo Cimento B*, 45:76–83, 1966.

[122] P. T. Landsberg. Special relativistic thermodynamics. *Proc. Phys. Soc.*, 89:1007–1016, 1966.

[123] T. W. B. Kibble. Relativistic transformation laws for thermodynamic variables. *Nuovo Cimento B*, 41:72–85, 1966.

[124] R. K. Pathria. Lorentz transformation of thermodynamic quantities. *Proc. Phys. Soc. (London)*, 88:791–799, 1966.

[125] R. K. Pathria. Lorentz transformation of thermodynamic quantities. II. *Proc. Phys. Soc. (London)*, 91:1–7, 1966.

[126] P. T. Landsberg. Does a moving body appear cool? *Nature*, 212:571–572, 1966.

[127] J. H. Fremlin. Does a moving body appear cool? *Nature*, 213:277, 1967.

[128] P. T. Landsberg. Does a moving body appear cool? *Nature*, 214:903–904, 1967.

[129] P. D. Noerdlinger. A moving body must "appear" cool. *Nature*, 213:1117, 1967.

[130] I. P. Williams. Temperature of a moving body. *Nature*, 213:1118, 1967.

[131] I. P. Williams. Does a moving body appear cooler. *Nature*, 214:1105, 1967.

[132] J. L. Redding. Temperature of a moving body. *Nature*, 215:1160–1161, 1967.

[133] C. Møller. Relativistic Thermodynamics. A Strange Incident in The History of Physics. *Kgl. Dan. Vidensk. Selsk., Mat.-Fys. Medd.*, 36(1):1–27, 1967.

[134] J. Lindhard. Temperature in special relativity. *Physica*, 38(4):635–640, 1968.

[135] A. G. Georgiu. Special Relativity and Thermodynamics. *Proc. Cambridge Phil. Soc.*, 66:423–429, 1966.

[136] P. T. Landsberg and K. A. Johns. The Problem of Moving Thermometers. *Proc. Roy. Soc. (London)*, 306(1478):477–486, 1968.

[137] N. G. van Kampen. Relativistic thermodynamics of moving systems. *Phys. Rev.*, 173:295–301, 1968.

[138] N. G. van Kampen. Lorentz-invariance of the distribution in phase space. *Physica*, 43:244–262, 1969.

[139] N. G. van Kampen. Relativistic thermodynamics. *J. Phys. Soc. Jap. Suppl.*, 26:316–321, 1969.

[140] N. G. van Kampen. A model for relativistic heat transport. *Physica*, 46:315–332, 1970.

[141] I. Müller. Toward relativistic thermodynamics. *Arch. Rat. Mech. Anal.*, 34:259–282, 1969.

[142] J. L. Lebowitz, N. G. van Kampen, J. E. Mayer, and J. L. Jackson. Discussion of relativistic thermodynamics. *J. Phys. Soc. Jap. Suppl.*, 26:321, 1969.

[143] V. H. Hamity. Relativistic thermodynamics. *Phys. Rev.*, 187:1745–1752, 1969.

[144] G. Horwitz. Rest Frames in Relativistic Thermodynamics. *Phys. Rev. D*, 4(12):3812–3813, 1971.

[145] H. Callen and G. Horwitz. Relativistic Thermodynamics. *Am. J. Phys.*, 39:938–947, 1971.

[146] P. T. Landsberg and K. A. Johns. The Lorentz Transformation of Heat and Work. *Ann. Phys.*, 56:299–318, 1970.

[147] A. Jabs. Lorentz-invariant and non-invariant momentum space and thermodynamics. *Nucl. Phys. B*, 34(1):177–188, 1971.

[148] Ø. Grøn. Asynchronous formulation of relativistic statics and thermodynamics. *Nuovo Cimento B*, 17(1):141–165, 1973.

[149] D. Eimerl. On relativistic thermodynamics. *Ann. Phys.*, 91(2):481–498, 1975.

[150] W. Israel. Nonstationary irreversible thermodynamics: A causal relativistic theory. *Ann. Phys.*, 100(1-2):310–331, 1976.

[151] N. Agmon. Relativistic transformations of thermodynamic quantities. *Found. Phys.*, 7(5-6):331–339, 1977.

[152] J. E. Krizan. Temperature and relativistic thermodynamics. *Phys. Lett. A*, 71(2-3):174–176, 1979.

[153] W. Israel and J. M. Stewart. Transient relativistic thermodynamics and kinetic theory. *Ann. Phys.*, 118(2), 1979.

[154] P. T. Landsberg. Thought Experiment to Determine the Special Relativistic Temperature Transformation. *Phys. Rev. Lett.*, 45(3):149–150, 1980.

[155] P. T. Landsberg. Einstein and statistical thermodynamics. I. Relativistic thermodynamics. *Eur. J. Phys.*, 2:203–207, 1981.

[156] I-Shih Liu, I. Müller, and T. Ruggeri. Relativistic thermodynamics of gases. *Ann. Physics (N.Y.)*, 169:191–219, 1986.

[157] A. Komar. Relativistic Temperature. *Gen. Rel. Grav.*, 27(11):1185–1206, 1995.

[158] P. T. Landsberg and G. E. A. Matsas. Laying the ghost of the relativistic temperature transformation. *Phys. Lett. A*, 223:401–403, 1996.

[159] G. M. Kremer and I. Müller. Dynamic pressure in relativistic thermodynamics. *Ann. Inst. Henri Poincare*, 67:111–121, 1997.

[160] J. Rembielinski, K. A. Smolinski, and G. Duniec. Thermodynamics and preferred frame. *Found. Phys. Lett.*, 14:487–500, 2001.

[161] C. K. Yuen. Lorentz transformation of thermodynamic quantities. *Am. J. Phys.*, 38:246–252, 1970.

[162] D. Ter Haar and H. Wegeland. Thermodynamics and statistical mechanics on the special theory of relativity. *Phys. Rep.*, 1(2):31–54, 1971.

[163] G. Neugebauer. *Relativistische Thermodynamik*. Reihe Wissenschaft. Friedr. Vieweg & Sohn, Wiesbaden/Braunschweig, 1981.

[164] C. Liu. Is There A Relativistic Thermodynamics? A Case Study in the Meaning of Special Relativity. *Stud. Hist. Phil. Sci.*, 25:983–1004, 1994.

[165] I. Müller. Speeds of propagation in classical and relativistic extended thermodynamics. *Living Reviews in Relativity*, 2(1), 1999. http://www.livingreviews.org/lrr-1999-1.

[166] K. von Mosengeil. Theorie der stationären Strahlung in einem gleichförmig bewegten Hohlraum. *Ann. Phys. (Leipzig)*, 22:867–904, 1907.

[167] F. Jüttner. Das Maxwellsche Gesetz der Geschwindigkeitsverteilung in der Relativtheorie. *Ann. Phys. (Leipzig)*, 34(5):856–882, 1911.

[168] J. C. Maxwell. On the dynamical theory of gases. *Phil. Trans. Roy. Soc. London*, 157:49–88, 1867.

[169] J. L. Synge. *The Relativistic Gas*. North-Holland, Amsterdam, 1957.

[170] F. Jüttner. Die relativistische Quantentheorie des idealen Gases. *Z. Phys.*, 47:542–566, 1928.

[171] R. C. Tolman. A General Theory of Energy Partition with Applications to Quantum Theory. *Phys. Rev.*, 11(4):261–275, 1918.

[172] H. Einbinder. Generalized Virial Theorems. *Phys. Rev.*, 74(7):803–805, 1948.

[173] John E. Krizan. Matter and radiation in equilibrium. *Phys. Rev. D*, 31(8):1968–1971, 1985.

[174] W. Lucha and F. F. Schöberl. Relativistic virial theorem. *Phys. Rev. Lett.*, 64(23):2733–2735, 1990.

[175] A. Komar. Relativistic Equipartition. *Gen. Rel. Grav.*, 28(4):379–385, 1995.

[176] P. G. Bergmann. Generalized statistical mechanics. *Phys. Rev.*, 84(5):1026–1033, Dec 1951.

[177] A. E. Scheidegger and R. V. Krotkov. Relativistic statistical thermodynamics. *Phys. Rev.*, 89(5):1096–1100, Mar 1953.

[178] J. E. Krizan and P. Havas. Relativistic Corrections in the Statistical Mechanics of Interacting Particles. *Phys. Rev.*, 128(6):2916–2924, 1962.

[179] N. A. Chernikov. Equilibrium distribution of the relativistic gas. *Acta Phys. Pol.*, 26:1069–1092, 1964.

[180] C. Møller. Gibbs' Statistical Mechanics in the Theory of Relativity. *Kgl. Dan. Vidensk. Selsk., Mat.-Fys. Medd.*, 36(16):1–44, 1968.

[181] J. H. Eberly and A. Kujawski. Relativistic Statistical Mechanics and Blackbody Radiation. *Phys. Rev.*, 155:10–19, 1967.

[182] R. Balescu. On the statistical mechanics of a relativistic quantum plasma. *Physica*, 31(11):1599–1602, 1965.

[183] R. Balescu. Velocity inversion in statistical mechanics. *Physica*, 36:433–456, 1967.

[184] R. Balescu and T. Kotera. On the covariant formulation of classical relativistic statistical mechanics. *Physica*, 33(3):558–580, 1967.

[185] R. Balescu, T. Kotera, and E. Piña. Lorentz transformations in phase space and in physical space. *Physica*, 33(3):581–594, 1967.

[186] R. Hakim. Relativistic perfect gas in an external force field. *Phys. Rev.*, 162:128–133, 1967.

[187] R. Balescu. Relativistic Statistical Thermodynamics. *Physica*, 40(3):309–338, 1968.

[188] R. Balescu. On relativistic statistical thermodynamics. *J. Phys. Soc. Japan Suppl.*, 26:313–315, 1969.

[189] R. Balescu and L. Brenig. Relativistic covariance of non-equilibrium statistical mechanics. *Physica*, 54(4):504–521, 1971.

[190] S. Nakajima. On relativistic statistical thermodynamics. *Prog. Theor. Phys.*, 41(6):1450–1460, 1968.

[191] A. G. Georgiu. A Relativistic Form of Statistical Thermodynamics. *Int. J. Theor. Phys.*, 3(1):41–45, 1970.

[192] R. Hakim. Remarks on Relativistic Statistical Mechanics. I. *J. Math. Phys.*, 8(6):1315–1344, 1965.

[193] R. Hakim. Remarks on Relativistic Statistical Mechanics. II. Hierarchies for the Reduced Densities. *J. Math. Phys.*, 8(7):1397–1400, 1965.

[194] R. Hakim. Sur la mécanique statistique relativiste. *Ann. Inst. Henri Poincaré (A)*, 6(3):225–244, 1967.

[195] E. A. Remler. Some connections between relativistic classical mechanics, statistical mechanics, and quantum field theory. *Phys. Rev. D*, 16(12):3464–3473, 1977.

[196] C. Cercignani. Speed of Propagation of Infinitesimal Disturbances in a Relativistic Gas. *Phys. Rev. Lett.*, 50(15):1122–1124, 1983.

[197] U. Ben Ya'acov. Lorentz symmetry of subdynamics in relativistic systems. *Physica A*, 222:307–329, 1995.

[198] T. Matolsci, J. Kristóf, and M. Székely. On the Momentum Distribution of Molecules of an Ideal Gas. *Publ. Appl. Analysis*, 7:1–14, 1996.

[199] V. Tretyak. On relativistic models in the equilibrium statistical mechanics. *Cond. Mat. Phys.*, 1:553–568, 1998.

[200] A. Duviryuk, A. Nazarenko, and V. Tretyak. Classical relativistic system of N charges. Hamiltonian description, form of dynamics, and partition function. *Cond. Mat. Phys.*, 4:5–14, 2001.

[201] F. Debbasch, J. P. Rivet, and W. A. van Leeuwen. Invariance of the relativistic one-particle distribution function. *Physica A*, 301:181–195, 2001.

[202] M. Montesinos and C. Rovelli. Statistical mechanics of generally covariant quantum theories: a boltzmann-like approach. *Class. Quantum Grav.*, 18:555–569, 2001.

[203] G. Kaniadakis. Statistical mechanics in the context of special relativity. *Phys. Rev. E*, 66:056125, 2002.

[204] C. Beck. Generalized statistical mechanics of cosmic rays. *Physica A*, 331(1-2):173–181, 2004.

[205] G. Ares de Parga, B. López-Carrera, and F. Angulo-Brown. A proposal for relativistic transformations in thermodynamics. *J. Phys A: Math. Gen.*, 38:2821–2834, 2005.

[206] W. C. Schieve. Covariant Relativistic Statistical Mechanics of Many Particles. *Found. of Physics*, 35(8):1359 – 1381, 2005.

[207] R. Silva and J. A. S. Lima. Relativity, nonextensivity and extended power law distributions. *Phys. Rev. E*, 72:057101, 2005.

[208] G. Kaniadakis. Statistical mechanics in the context of special relativity. ii. *Phys. Rev. E*, 72:036108, 2005.

[209] G. Kaniadakis. Towards a relativistic statistical theory. *Physica A*, 365:17–23, 2006.

[210] R. Silva. The relativistic statistical theory and Kaniadakis entropy: an approach through a molecular chaos hypothesis. *Eur. Phys. J. B*, 54(4):499–502, 2006.

[211] E. Lehmann. Covariant equilibrium statistical mechanics. *J. Math. Phys.*, 47:023303, 2006.

[212] F. Becattini and L. Ferroni. The microcanonical ensemble of the ideal relativistic quantum gas. arXiv:nucl-th/0704.1967v1.

[213] G. Lina, D. Jiulin, and L. Zhipeng. The property of kappa-deformed statistics for a relativistic gas in an electromagnetic field: kappa parameter and kappa-distribution. arXiv:0704.1437v1.

[214] A. Aliano, L. Rondoni, and G. P. Moriss. Maxwell-Jüttner distributions in relativistic molecular dynamics. *Eur. Phys. J. B*, 50:361–365, 2006.

[215] L. P. Horwitz, W. C. Schieve, and C. Piron. Gibbs Ensembles in Relativistic Classical and Quantum Mechanics. *Ann. Phys.*, 137:306–340, 1981.

[216] L. P. Horwitz, S. Shashoua, and W. C. Schieve. A manifestly covariant relativistic Boltzmann equation for the evolution of a system of events. *Physica A*, 161:300–338, 1989.

[217] L. Burakovsky and L. P. Horwitz. Generalized Boltzmann equation in a manifestly covariant relativistic statistical mechanics. *Found. of Physics*, 25(9):1335 – 1358, 1995.

[218] B. Cleuren, K. Willaert, A. Engel, and C. Van den Broeck. Fluctuation theorem for entropy production during effusion of a relativistic ideal gas. *Phys. Rev. E*, 77(2):022103, 2008.

[219] F. Debbasch. Equilibrium distribution function of a relativistic dilute perfect gas. *Physica A*, 387(11):2443–2454, 2007.

[220] R. Becker. *Theory of heat.* Springer, New York, 1967.

[221] K. Huang. *Statistical Mechanics.* Wiley, NewYork, 1963.

[222] J. A. Wheeler and R. P. Feynman. Interaction with the Absorber as the Mechanism of Radiation. *Rev. Mod. Phys.*, 17(2-3):157–181, 1945.

[223] J. A. Wheeler and R. P. Feynman. Classical Electrodynamics in Terms of Direct Interparticle Action. *Rev. Mod. Phys.*, 21(3):425–433, 1949.

[224] H. Van Dam and E. P. Wigner. Classical Relativistic Mechanics of Interacting Point Particles. *Phys. Rev.*, 138(6B):B1576–B1582, 1965.

[225] H. Van Dam and E. P. Wigner. Instantaneous and Asymptotic Conservation Laws for Classical Relativistic Mechanics of Interacting Point Particles. *Phys. Rev.*, 142(4):838–843, 1966.

[226] D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan. Relativistic Invariance and Hamiltonian Theories of Interacting Particles. *Rev. Mod. Phys.*, 35(2):350–375, 1963.

[227] H. Leutwyler. A no Interaction Theorem in Classical Relativistic Hamiltonian Particle Mechanics. *Nuovo Cimento*, 37:556–567, 1965.

[228] G. Marmo, N. Mukunda, and E. C. G. Sudarshan. Relativistic particle dynamics – Lagrangian proof of the no-interaction theorem. *Phys. Rev. D*, 30(10):2110–2116, 1984.

[229] A. D. Fokker. Ein invarianter Variationssatz für die Bewegung mehrerer elektrischer Massenteilchen. *Z. Physik A: Hadrons and Nuclei*, 58(5-6):386–393, 1929.

[230] M. H. L. Pryce. The mass-centre in the restricted theory of relativity and its connexion with the quantum theory of elementary particles. *Proc. Roy. Soc. London*, 195(1040):62–81, 1948.

[231] P. Havas. Four-Dimensional Formulations of Newtonian Mechanics and Their Relation to the Special and the General Theory of Relativity. *Rev. Mod. Phys.*, 36(4):938–965, 1964.

[232] P. Havas and J. N. Goldberg. Lorentz-Invariant Equations of Motion of Point Masses in the General Theory of Relativity. *Phys. Rev.*, 128(1):398–414, 1962.

[233] R. Arens and D. G. Babbitt. The geometry of relativistic n-particle interactions. *Pacific J. Math.*, 28(2), 1969.

[234] C. K. Chou. The vlasov equations. Technical Report NASA-CR-74673; E20-151, 19651101, NASA Center for AeroSpace Information (CASI), 1965. available from `http://ntrs.nasa.gov/`.

[235] P. Havas and J. Stachel. Invariances of approximately relativistic lagrangians and the center-of-mass theorem. i. *Phys. Rev.*, 185(5):1636–1647, 1969.

[236] H. W. Woodcock and P. Havas. Approximately Relativistic Lagrangians for Classical Interacting Point Particles. *Phys. Rev. D*, 6(12):3422–3444, 1972.

[237] E. H. Kerner, editor. *The Theory of Action-at-a-Distance in Relativistic Particle Dynamics. A Reprint Collection.*, volume 11 of *International science review series*. Gordon and Breach, New York, 1972.

[238] P. Droz-Vincent. Relativistic systems of interacting particles. *Phys. Scr.*, 2:129–134, 1970.

[239] T. Gill and J. Lindesay. Canonical proper time formulation of relativistic particle dynamics. *Int. J. Theor. Phys.*, 32(11):2087–2098, 1993.

[240] P. Ramond. Action-at-a - Distance Theories and Dual Models. *Phys. Rev. D*, 7(2):449–458, 1973.

[241] R. Marnelius. Lagrangian and Hamiltonian formulation of relativistic particle mechanics. *Phys. Rev. D*, 10(8):2535–2553, 1974.

[242] R. P. Gaida, Yu. B. Klyuchkovskii, and V. I. Tretyak. Lagrangian classical relativistic mechanics of a system of directly interacting particles. I. *Theor. Math. Phys.*, 44(2):687–697, 1980.

[243] R. P. Gaida, Yu. B. Klyuchkovskii, and V. I. Tretyak. Lagrangian classical relativistic mechanics of a system of directly interacting particles. II. *Theor. Math. Phys.*, 45(2):963–975, 1980.

[244] R. P. Gaida, Yu. B. Klyuchkovskii, and V. I. Tretyak. Forms of relativistic dynamics in a classical Lagrangian description of a system of particles. *Theor. Math. Phys.*, 55(1):372–384, 1983.

[245] S. N. Sokolov and V. I. Tretyak. Front form of relativistic Lagrangian dynamics in two-dimensional space time and its connection with the Hamiltonian description. *Theor. Math. Phys.*, 67(1):385–394, 1986.

[246] H. Sorge, H. Stöcker, and W. Greiner. Poincare-Invariant Hamiltonian Dynamics: Modelling Multi-Hadronic Interactions in a Phase Space Approach. *Ann. Phys.*, 192:266–306, 1989.

[247] R. P. Gaida, V. I. Tretyak, and Yu. G. Yaremko. Center-of-mass variables in the relativistic Lagrangian dynmics of a system of particles. *Theor. Math. Phys.*, 101(3):1443–1453, 1994.

[248] P. A. M. Dirac. Forms of Relativistic Dynamics. *Rev. Mod. Phys.*, 21(3):392–399, 1949.

[249] J. L. Anderson and P. G. Bergmann. Constraints in Covariant Field Theories. *Phys. Rev.*, 83(5):1018–1025, 1951.

[250] I. T. Todorov. Dynamics of relativistic point particles as a problem with constraints. JINR-E2-10125, 1976. 26pp.

[251] A. Komar. Constraint formalism of classical mechanics. *Phys. Rev. D*, 18(6):1881–1886, 1978.

[252] A. Komar. Interacting relativistic particles. *Phys. Rev. D*, 18(6):1887–1893, 1978.

[253] J. N. Goldberg, E. C. G. Sudarshan, and N. Mukunda. Relativistically interacting particles and world lines. *Phys. Rev. D*, 23(10):2231–2235, 1981.

[254] A. Kihlberg, R. Marnelius, and N. Mukunda. Relativistic potential models as systems with constraints and their interpretation. *Phys. Rev. D*, 23(10):2201–2209, 1981.

[255] F. Rohrlich. Many-body forces and the cluster decomposition. *Phys. Rev. D*, 23(6):1305–1315, 1981.

[256] F. Rohrlich. Evolution and covariance in constraint dynamics. *Phys. Rev. D*, 25(10):2576–2583, 1982.

[257] J. Samuel. Constraints in relativistic Hamiltonian mechanics. *Phys. Rev. D*, 26(12):3475–3481, 1982.

[258] L. P. Horwitz and F. Rohrlich. Constraint relativistic quantum dynamics. *Phys. Rev. D*, 24(6):1528–1542, 1981.

[259] L. P. Horwitz and F. Rohrlich. Limitations of constraint dynamics. *Phys. Rev. D*, 31(4):932–933, 1985.

[260] F. Marquès, V. Iranzo, A. Molina, A. Montoto, and J. Llosa. World-line condition and the noninteraction theorem. *Phys. Rev. D*, 31(2):314–318, 1985.

[261] G. Longhi and L. Lusanna. Bound-state solutions, invariant scalar products, and conserved currents for a class of two-body relativistic systems. *Phys. Rev. D*, 34(12):3707–3731, 1986.

[262] M. Montesinos, C. Rovelli, and T. Thiemann. $SL(2, R)$ model with two Hamiltonian constraints. *Phys. Rev. D*, 60(4):044009, 1999.

[263] Cheuk-Yin Wong and Horace W. Crater. Relativistic N-body problem in a separable two-body basis. *Phys. Rev. C*, 63(4):044907, 2001.

[264] V. Gerdt, A. Khvedelidze, and Yu. Palii. Towards an algorithmisation of the dirac constraint formalism. In J. Calmet, W. M. Seiler, and R. W. Tucker, editors, *Global Integrability of Field Theories. Proceedings of GIFT 2006.*, pages 135–154, Daresbury, UK, 2006. Cocroft Institute.

[265] R. Hakim. Statistical mechanics of relativistic dense matter systems. *Riv. Nuovo Cim.*, 1(6):1–52, 1978.

[266] R. Hakim and H. D. Sivak. Relativistic statistical mechanics, a brief overview. *AIP Conf. Proc.*, 841:63–99, 2006. A Century of Relativity Physics: ERE 2005; XXVIII Spanish Relativity Meeting.

[267] M. H. Ernst. Nonlinear Model-Boltzmann Equations and Exact Solutions. *Phys. Rep.*, 78(1):1–171, 1981.

[268] R. L. Liboff. *Kinetic Theory*. Prentice Hall, Inc., Englewood Cliffs, New Jersey 07632, 1990.

[269] C. Eckart. The Thermodynamics of Irreversible Processes. III. Relativistic Theory of the Simple Fluid. *Phys. Rev.*, 58(10):919–924, 1940.

[270] A. Lichnerowicz and R. Marrot. Propriétés statistiques des ensembles de particules en relativité. *Compt. Rend. Acad. Sci. Paris*, 210:795–761, 1940.

[271] G. A. Kluitenberg, S. R. de Groot, and P. Mazur. Relativistic thermodynamics of irreversible processes. I. Heat conduction, diffusion, viscous flow and chemical reactions; formal part. *Physica*, 19:689–704, 1953.

[272] S. T. Beliaev and G. I. Budker. Relativistic kinetic equation. *Dokl. Akad. Nauk SSSR*, 6:807–810, 1956. English transl.: Soviet Phys. Doklady 1, 218 (1956).

[273] W. Israel. Relativistic kinetic theory of a simple gas. *J. Math. Phys.*, 4(9):1163–1181, 1963.

[274] Yu. L. Klimontovich. *The Statistical Theory of Non-Equilibrium Processes in a Plasma*. MIT Press, Cambridge MA, 1967.

[275] A. Mangeney. Equations cinetiques pour un plasma homogene en interaction avec un champ de rayonnement. *Physica*, 30(3):461–474, 1964.

[276] Ph. de Gottal and I. Prigogine. Relativistic effects in statistical hydrodynamics. *Physica*, 31(5):677–687, 1965.

[277] R. W. Lindquist. Relativistic transport theory. *Ann. of Phys.*, 37:487–518, 1966.

[278] H. Akama. Relativistic Boltzmann equation for plasmas. *J. Phys. Soc. Japan*, 28(2):478–488, 1970.

[279] S. R. de Groot. On relativistic kinetic theory of transport processes, in particular of neutrino systems. *Ann. Inst. Henri Poincaré (A) Phys. théor.*, 31A:377–386, 1979.

[280] R. J. Gould. Kinetic theory of relativistic plasmas. *Phys. Fluids*, 24(1), 1981.

[281] D. Pavón, D. Jou, and J. Casas-Vázquez. On a covariant formulation of dissipative phenomena. *Ann. Inst. Henri Poincaré (A) Phys. théor.*, 36(1):79–88, 1982.

[282] C. Cercignani and G. M. Kremer. On relativistic collisional invariants. *J. Stat. Phys.*, 96(1-2):439–445, 1999.

[283] C. Cercignani and G. M. Kremer. Trend to euilibrium of a degenerate relativistic gas. *J. Stat. Phys.*, 98(1-2):441–456, 2000.

[284] M. Lemou. Linearized quantum and relativistic fokker-planck-landau equations. *Math. Meth. Appl. Sci.*, 23(12):1093–1119, 2000.

[285] M. Honda. Relativistic heat conduction and thermoelectric properties of nonuniform plasmas. *Phys. Plasmas*, 10(10):4177–4180, 2003.

[286] R. M. Strain and Y. Guo. Stability of the Relativistic Maxwellian in a Collisional Plasma. *Comm. Math. Phys.*, 251(2):263–320, 2004.

[287] L. Hsiao and H. Yu. Asymptotic stability of the relativistic Maxwellian. *Math. Meth. Appl. Sci.*, 29(13):1481 – 1499, 2006.

[288] J. M. Stewart. *Non-Equilibrium Relativistic Kinetic Theory*, volume 10 of *Lecture Notes in Physics*. Springer, Berlin, 1971.

[289] S. R. de Groot, W. A. van Leeuwen, and C. G. van Weert. *Relativistic Kinetic Theory: Principles and Applications*. North-Holland, Amsterdam, 1980.

[290] C. Cercignani and G. M. Kremer. *The Relativistic Boltzmann Equation: Theory and Applications*, volume 22 of *Progress in mathematical physics*. Birkhäuser Verlag, Basel, Boston, Berlin, 2002.

[291] J. Ehlers. General relativity and kinetic theory. In R. K. Sachs, editor, *General Relativity and Cosmology*, pages 1–70. Academic Press, New York, 1971. Proc. of the Int. School of Physics "Enrico Fermi", Course XLVII.

[292] H. Andréasson. The Einstein-Vlasov System/Kinetic Theory. *Living Reviews in Relativity*, 8(2), 2005.

[293] G. Schay. *The equations of diffusion in the special theory of relativity*. PhD thesis, Princeton University, 1961. available through University Microfilms, Ann Arbor, Michigan, https://wwwlib.umi.com.

[294] G. Chacón-Acosta and G. M. Kremer. Fokker-planck-type equations for a simple gas and for a semirelativistic brownian motion from a relativistic kinetic theory. *Phys. Rev. E*, 76(2):021201, 2007.

[295] P. Danielewicz. Nonrelativistic and relativistic Landau/Fokker-Planck equations for arbitrary statistics. *Physica*, 100A:167–182, 1980.

[296] H. van Erkelens. Relativistic Boltzmann theory for a plasma. VI. The relativistic Landau equation. *Physica*, 107A:48–70, 1981.

[297] N. J. Fisch. Current generation in a relativistic plasma. *Phys. Rev. A*, 24(6):3245–3248, 1981.

[298] C. F. F. Karney and N. J. Fisch. Efficiency of current drive by fast waves. *Phys. Fluids*, 28(1):116–126, 1985.

[299] P. de Gottal and J. Gariel. Test-particle motion in a relativistic plasma. I. Weakly coupled medium. *Physica A*, 157:1059–1073, 1989.

[300] Ph. de Gottal and J. Gariel. Test-particle motion in a relativistic plasma. II. The Lorentz gas. *Physica A*, 163:615–624, 1990.

[301] Ph. de Gottal and J. Gariel. Test-particle motion in a relativistic plasma. III. Collective effects. *Physica A*, 186:534–548, 1992.

[302] O. V. Klimov and A. A. Tel'Nikhin. Stochastic dynamics of a relativistic particle in a wave-packet field. *Plasma Phys. Rep.*, 21:335–338, 1995.

[303] J. R. Davies, A. R. Bell, M. G. Haines, and S. M. Guérin. Short-pulse high-intensity laser-generated fast electron transport into thick solid targets. *Phys. Rev. E*, 56(6):7193–7203, 1997.

[304] K. C. Baral and J. N. Mohanty. Covariant formulation of the Fokker-Planck equation for moderately coupled relativistic magnetoplasma. *Phys. Plasmas*, 7:1103–1112, 2000.

[305] K. C. Baral and J. N. Mohanty. Relativistic theory of moderately coupled plasmas: Formulation of appropriate fokker-planck equation. *Phys. Plasmas*, 9(11):4507–4511, 2002.

[306] T. Yokota, Y. Nakao, T. Johzaki, and K. Mima. Two-dimensional relativistic Fokker-Planck model for core plasma heating in fast ignition targets. *Phys. Plasmas*, 13(1):022702, 2006.

[307] G. V. Khazanov, A. A Tel'nikhina, and T. K. Kronberg. Chaotic motion of relativistic electrons driven by whistler waves. *Plasma Phys. Control. Fusion*, 49(4):447–466, 2007.

[308] B. Svetitsky. Diffusion of charmed quarks in the quark-gluon plasma. *Phys. Rev. D*, 37(9):2484–2491, 1988.

[309] P. Roy, J. E. Alam, S. Sarkar, B. Sinha, and S.Raha. Quark-gluon plasma diagnostics in a successive equilibrium scenario. *Nucl. Phys. A*, 624(4):687–705, 1997.

[310] H. van Hees and R. Rapp. Thermalization of heavy quarks in the quark-gluon plasma. *Phys. Rev. C*, 71(3):034907, 2005.

[311] G. Wolschin, M. Biyajima, T. Mizoguchi, and N. Suzuki. Time evolution of relativistic d+au and au+au collisions. *Ann. Phys.*, 15(6):369–378, 2006.

[312] H. van Hees, V. Greco, and R. Rapp. Heavy-quark probes of the quark-gluon plasma and interpretation of recent data taken at the BNL Relativistic Heavy Ion Collider. *Phys. Rev. C*, 73:034913, 2006.

[313] H. van Hees, V. Greco, and R. Rapp. Thermalization and flow of heavy quarks in the quark-gluon plasma. *AIP Conf. Proc.*, 842:77–79, 2006.

[314] R. Rapp, V. Greco, and H. van Hees. Heavy-Quark Spectra at RHIC and Resonances in the QGP. *Nucl. Phys. A*, 774:685–688, 2006.

[315] P. J. D. Mauas and D. O. Gomez. Kinetic description of electron beams in the solar chromosphere. *Astophys. J.*, 398:682–691, 1997.

[316] P. J. D. Mauas and D. O. Gomez. Fokker-Planck Description of Electron Beams in the Solar Chromosphere. *Astophys. J.*, 483:496–506, 1997.

[317] N. Itoh, Y. Kohyama, and S. Nozawa. Relativistic corrections to the Sunyaev-Zeldovich effect for clusters of galaxies. *Astrophys. J.*, 502:7–15, 1998.

[318] B. Wolfe and F. Melia. Covariant kinetic theory with an application to the coma cluster. *Astrophys. J.*, 638, 2006.

[319] P. A. Becker, T. Le, and C. D. Dermer. Time-dependent stochastic particle acceleration in astrophysical plasmas: Exact solutions including momentum-dependent escape. *Astrophys. J.*, 647(1):539–551, 2006.

[320] E. Bertschinger. Brownian Motion of Stars, Dust, and Invisible Matter. *AIP Conf. Proc.*, 861(1):97–105, 2006.

[321] C. F. F. Karney. Fokker-Planck and quasilinear codes. *Comp. Phys. Rep.*, 4(3-4):183–244, 1986.

[322] M. Shoucri, V. Fuchs, and A. Bers. The application of the finite-element library TWODEPEP for the numerical solution of the relativistic Fokker-Planck equation. *Comp. Phys. Comm.*, 46(3):337–344, 1987.

[323] Y. Peysson and M. Shoucri. An approximate factorization procedure for solving nine-point elliptic difference equations - Application for a fast 2-D relativistic Fokker-Planck solver. *Comp. Phys. Comm.*, 109(1):55–80, 1998.

[324] T. H. Boyer. Inconsistency between the Boltzmann distribution for relativistic free particles and the Rayleigh-Jeans law for thermal radiation within classical electrodynamics. *Phys. Rev. D*, 19:1112–1115, 1979.

[325] T. H. Boyer. Inconsistency between the Boltzmann distribution for relativistic free particles and the Planck spectrum for thermal radiation in quantum theory. *Phys. Rev. D*, 19:3635–3640, 1979.

[326] U. Ben-Ya'acov. Relativistic Brownian motion and the spectrum of thermal radiation. *Phys. Rev. D*, 23(6):1441–1450, 1981.

[327] R. Blanco, L. Pesquera, and E. Santos. Equilibrium between radiation and matter for classical relativistic multiperiodic systems. Derivation of Maxwell-Boltzmann distribution from Rayleigh-Jeans spectrum. *Phys. Rev. D*, 27(6):1254–1287, 1983.

[328] C. D. Dermer. The production spectrum of a relativistic Maxwell-Boltzmann gas. *Astrophys. J.*, 280:328–333, 1984.

[329] O. Oron and L. P. Horwitz. Relativistic Brownian Motion and Gravity as an Eikonal Approximation to a Quantum Evolution Equation. *Found. Phys.*, 35(7):1181 – 1203, 2005.

[330] F. Castejón and S. Eguilior. Langevin equations for quasi-linear wave–particle interaction. *Plasma Phys. Control. Fusion*, 45(2):159–167, 2003.

[331] K. S. Fa. Analysis of the relativistic brownian motion in momentum space. *Braz. J. Phys.*, 36(3A):777–782, 2006.

[332] A. Cabo-Bizet and A. Cabo Montes de Oca. Particle dynamics in a relativistic invariant stochastic medium. *Phys. Lett. A*, 359:265–271, 2006.

[333] R. Zygadło. Two Markovian models leading to the Pascalian (1D ultrarelativistic) stationary distribution. *Phys. Lett. A*, 329:459–463, 2004.

[334] R. Zygadło. Free Brownian motion approach to the ideal gas of relativistic particles. *Phys. Lett. A*, 345:323–329, 2005.

[335] C. Chevalier and F. Debbasch. Relativistic diffusions: A unifying approach. *J. Math. Phys.*, 49:043303, 2008.

[336] P. E. Kloeden and E. Platen. *Numerical Solution of Stochastic Differential Equations*, volume 23 of *Stochastic Modelling and Applied Probability*. Springer, 2006.

[337] R. Rapp and H. van Hees. Heavy quark diffusion as a probe of the quark-gluon plasma. arXiv:0803.0901v1, 2008.

[338] M. E. Dieckmann, L. O'C. Drury, and P. K. Shukla. On the ultrarelativistic two-stream instability, electrostatic turbulence and Brownian motion. *New J. Phys.*, 8:40, 2006.

[339] J. Łopuszańnski. Relativisierung der theorie der stochastischen prozesse. *Acta Phys. Polon.*, 12:87–99, 1953.

[340] H. Rudberg. *On the Theory of Relativistic Diffusion*. Almquist and Wiksells, Uppsala, 1957.

[341] R. M. Dudley. Lorentz-invariant Markov processes in relativistic phase space. *Ark. Mat. Astron. Fys.*, 6:241–268, 1965.

[342] R. M. Dudley. A note on Lorentz-invariant Markov processes. *Ark. Mat. Astron. Fys.*, 6:575–581, 1967.

[343] R. M. Dudley. Asymptotics of Some Relativistic Markov Processes. *Proc. Nat. Acad. Sci. USA*, 70(12):3551–3555, 1973.

[344] R. M. Dudley. Recession of some relativistic Markov processes. *Rocky Mountain J. Math.*, 4:401–406, 1967.

[345] R. Hakim. A covariant theory of relativistic Brownian Motion. I. Local Equilibrium. *J. Math. Phys.*, 6(10):1482–1495, 1965.

[346] R. Hakim. Relativistic stochastic processes. *J. Math. Phys.*, 9:1805–1818, 1968.

[347] E. Nelson. Derivation of the Schrödinger Equation from Newtonian Mechanics. *Phys. Rev.*, 150(4):1079–1085, 1966.

[348] K. Yasue. Derivation of Relativistic Wave Equations in the Theory of Elementary Domains. *Prog. Theor. Phys.*, 57(1):318–328, 1977.

[349] J.-P. Caubet. Relativistic Brownian motion. In *Proceedings of the Conference on Probabilistic Methods in Differential Equations*, number 451 in Lecture Notes in Mathematics, pages 131–142, Berlin, 1975. University of Victoria, 1974, Springer.

[350] J. P. Caubet. Mouvement brownien spatiotemporal non isotrope. *Compt. Rend. Acad. Sci. Paris*, 280:303–306, 1975.

[351] J. P. Caubet. Relativistic Brownian motion. *Adv. Appl. Prob.*, 8(2):230–231, 1976.

[352] J. P. Caubet. *Le Mouvement Brownian Relativiste*. Number 559 in Lecture Notes in Mathematics. Springer, Berlin, Heidelberg, 1976.

[353] E. Nelson. Review of 'Le Mouvement Brownian Relativiste ' by J. P. Caubet. *Bull. American Math. Soc.*, 84:121–124, 1978.

[354] J. P. Caubet. Brownian Motion, Minkowski Space and Principle of Special Relativity. *Comp. Rend. Sci. Acad. Serie A*, 284(5):339–341, 1977.

[355] P. Suppes. Probability in relativistic particle theory. *Erkenntnis*, 16(2):299–305, 1981.

[356] F. Guerra and P. Ruggiero. A note on relativistic Markov processes. *Lett. Nouvo Cimento*, 23:529–534, 1978.

[357] F. Guerra and P. Ruggiero. New Interpretation of the Euclidean-Markov Field in the Framework of Physical Minkowski Space-Time. *Phys. Rev. Lett.*, 31(16):1022–1025, 1978.

[358] K. Namsrai. Relativistic dynamics of stochastic particles. *Found. Phys.*, 10(3-4):353–361, 1980.

[359] S. Roy. Relativistic Brownian motion and the space–time approach to quantum mechanics. *J. Math. Phys.*, 21:71–76, 1980.

[360] S. Roy. Relativistic Brownian motion and the theory of measurement in quantum mechanics. *Nuovo Cim. B*, 64:81–93, 1981.

[361] J. P. Caubet. The Brownian Theory of Light. *Stoch. Anal. Appl.*, 2(1):55–85, 1984.

[362] J. P. Caubet. The great fugue of the Brownian theory of light. *Stoch. Anal. Appl.*, 3(2):119–151, 1985.

[363] E. Nelson. Clock space. In F. Guerra, M. I. Loffredo, and C. Marchioro, editors, *Probabilistic Methods in Mathematical Physics*, pages 329–333. World Scientific, Singapore, 1992.

[364] P. Garbaczewski. Relativistic Problem of Random Flights and Nelson Stochastic Mechanics. *Phys. Lett. A*, 164(1):6–16, 1992.

[365] L. M. Morato and L. Viola. Markov diffusions in comoving coordinates and stochastic quantization of the free relativistic particle. *J. Math. Phys.*, 36:4691–4710, 1995. Erratum, *ibid.* **37**, 4769 (1996).

[366] P. Garbaczewski, J. R. Klauder, and R. Olkiewicz. Schrödinger problem, Lévy processes, and noise in relativistic quantum mechanics. *Phys. Rev. E*, 51(5):4114–4131, 1995.

[367] M. Nagasawa. Quantum theory, theory of Brownian motions, and relativity theory. *Chaos, Solitons & Fract.*, 7(5):631–643, 1996.

[368] A. Posilicano. Poincareé invariant Markov processes and Gaussian random fields on relativistic phase-spaces. *Lett. Math. Phys.*, 42:85–93, 1997.

[369] M. Pavon. On the stochastic mechanics of the free relativistic particle. *J. Math. Phys.*, 42(10):4946–4856, 2001.

[370] J. R. Fanchi. Review of Invariant Time Formulations of Relativistic Quantum Theories. *Found. Phys.*, 23(3):487–548, 1993.

[371] H. Grabert, P. Hänggi, and P. Talkner. Is quantum mechanics equivalent to a classical stochastic process? *Phys. Rev. A*, 19(6):2440–2445, 1979.

[372] D. L. Rapoport. On the unification of geometric and random structures through torsion fields: Brownian motion, viscous and magneto-fluid-dynamics. *Found. Phys.*, 35(7):1205–1243, 2005.

[373] D. L. Rapoport. Cartan-Weyl Dirac and Laplacian Operators, Brownian Motions: The Quantum Potential and Scalar Curvature, Maxwell's and Dirac-Hestenes Equations, and Supersymmetric Systems. *Found. Phys.*, 35(8):1385–1431, 2005.

[374] J. Franchi and Y. Le Jan. Relativistic diffusions and Schwarzschild geometry. *Comm. Pure Appl. Math.*, 60:187–251, 2006.

[375] S. Goldstein. On diffusion by discontinuous movements, and on the telegraph equation. *Q. J. Mech. App. Math.*, 4(2):129–156, 1951.

[376] M. Kac. A stochastic model related to the telegrapher's equation. *Rocky Mountain J. Math.*, 4:497–509, 1974.

[377] J. Masoliver and G. H. Weiss. Finite-velocity diffusion. *Eur. J. Phys.*, 17:190–196, 1996.

[378] R. Fürth. Einige Untersuchungen über Brownsche Bewegungen an einem Einzelteilchen. *Ann. Phys. (Berlin)*, 53:177–213, 1917.

[379] R. Fürth. Die Brownsche Bewegung bei Berücksichtigung einer Persistenz der Bewegungsrichtung. Mit Anwendungen auf die Bewegung lebender Infusorien. *Z. Physik*, 2:244–256, 1920.

[380] G. I. Taylor. Diffusion by continuous movements. *Proc. London Math. Soc.*, 20:196–212, 1922.

[381] M. Boguñá, J. M. Porrà, and J. Masoliver. Generalization of the persistent random walk to dimensions greater than 1. *Phys. Rev. E*, 58(6):6992–6998, 1998.

[382] M. Boguñá, J. M. Porrà, and J. Masoliver. Persistent random walk model for transport through thin slabs. *Phys. Rev. E*, 59(6):6517–6526, 1999.

[383] A. Ishimaru. Diffusion of light in turbid material. *Appl. Opt.*, 28:2210–2215, 1989.

[384] M. Miri and H. Stark. Persistent random walk in a honeycomb structure: Light transport in foams. *Phys. Rev. E*, 68(3):031102, 2003.

[385] M. Miri, E. Madadi, and H. Stark. Diffusive transport of light in a Kelvin foam. *Phys. Rev. E*, 72(3):031111, 2005.

[386] M. Miri, Z. Sadjadi, and M. E. Fouladvand. Persistent random walk on a site-disordered one-dimensional lattice: Photon subdiffusion. *Phys. Rev. E*, 73(3):031115, 2006.

[387] H. Kim, C. Kim, E. K. Lee, P. Talkner, and P. Hänggi. Wall-mediated self-diffusion in slit and cylindrical pores. *Phys. Rev. E*, 77:031202, 2008.

[388] D. D. Joseph and L. Preziosi. Heat waves. *Rev. Mod. Phys.*, 61(1):41–73, 1989.

[389] D. D. Joseph and L. Preziosi. Addendum to the paper "Heat waves" [Rev. Mod. Phys. 61, 41 (1989)]. *Rev. Mod. Phys.*, 62(2):375–391, 1990.

[390] P. Kostädt and M. Liu. Causality and stability of the relativistic diffusion equation. *Phys. Rev. D*, 62:023003, 2000.

[391] P. Kostädt and M. Liu. Alleged acausality of the diffusion equations: A reply. *Phys. Rev. D*, 64:088504, 2001.

[392] L. Herrera and D. Pavón. Why hyperbolic theories of dissipation cannot be ignored: Comment on a paper by Kostädt and Liu. *Phys. Rev. D*, 64:088503, 2001.

[393] M. Abdel-Aziz and S. Gavin. Causal diffusion and the survival of charge fluctuations in nuclear collisions. *Phys. Rev. C*, 70(3):034905, 2004.

[394] M. Abdel-Aziz and S. Gavin. Causal diffusion at RHIC. *J. Phys. G: Nucl. Phys.*, 31:S77–S84, 2005.

[395] T. Koide. Microscopic derivation of causal diffusion equation using the projection operator method. *Phys. Rev. E*, 72:026135, 2005.

[396] T. Koide. Microscopic formula of transport coefficients for causal hydrodynamics. *Phys. Rev. E*, 75:060103(R), 2007.

[397] M. E. Peskin and D. V. Schroeder. *An Introduction to Quantum Field Theory.* Perseus Books, Reading, Mass., 1995.

[398] B. Gaveau, T. Jacobson, M. Kac, and L. S. Schulman. Relativistic Extension of the Analogy between Quantum Mechanics and Brownian Motion. *Phys. Rev. Lett.*, 53(5):419–422, 1984.

[399] T. Ichinose. Path integral for the Dirac equation in two space-time dimensions. *Proc. Japan Acad.*, 58:290–293, 1982.

[400] T. Ichinose. Path integral formulation of the propagator for a two-dimensional Dirac particle. *Physica A*, 124(1-3):419–425, 1984.

[401] T. Jacobson and L. S. Schulman. Quantum stochastics: the passage from a relativistic to a non-relativistic path integral. *J. Phys. A: Math. Gen.*, 17:375–383, 1984.

[402] A. O. Barut and I. H. Duru. Path-Integral Derivation of the Dirac Propagator. *Phys. Rev. Lett.*, 53(25):2355–2358, 1984.

[403] B. Gaveau and L. S. Schulman. Grassmann-valued processes for the Weyl and the Dirac equations. *Phys. Rev. D*, 36(4):1135–1140, 1987.

[404] R. P. Feynman and A. R. Hibbs. *Quantum Mechanics and Path Integrals.* McGraw-Hill, New York, 1965. problem 2-6, pp. 34-36.

[405] W. L. Kath. Waiting and propagating fronts in nonlinear diffusion. *Physica D*, 12:375–381, 1984.

[406] G. Wolschin. Diffusion and local deconfinement in relativistic systems. *Phys. Rev. C*, 69(2):024906, February 2004.

[407] C. W. Gardiner. *Handbook of Stochastic Methods*. Springer Series in Synergetics. Springer, Berlin, 2 edition, 2002.

[408] R. Benzi, A. Sutera, and A. Vulpiani. The mechanism of stochastic resonance. *J. Phys.*, A14:L453–L457, 1981.

[409] R. Benzi, G. Parisi, A. Sutera, and A. Vulpiani. A theory of stochastic resonance in climatic change. *SIAM J. Appl. Math.*, 43:565–578, 1983.

[410] C. Nicolis and G. Nicolis. Stochastic aspects of climatic transitions - additive fluctuations. *Tellus*, 33:225–234, 1981.

[411] C. Nicolis. Long-term climatic transitions and stochastic resonance. *J. Stat. Phys.*, 70(1/2):3–13, 1993.

[412] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni. Stochastic resonance. *Rev. Mod. Phys.*, 70(1):223–287, 1998.

[413] A. Neiman and L. Schimansky-Geier. Stochastic resonance in bistable systems driven by harmonic noise. *Phys. Rev. Lett.*, 72(19):2988–2991, 1994.

[414] V. S. Anishchenko, A. B. Neiman, F. Moss, and L. Shimansky-Geier. Stochastic resonance: noise-enhanced order. *Sov. Phys. Usp.*, 42:7–36, January 1999.

[415] B. Lindner and L. Schimansky-Geier. Coherence and stochastic resonance in a two-state system. *Phys. Rev. E*, 61(6):6103–6110, 2000.

[416] B. Lindner, J. Garca-Ojalvo, A. Neiman, and L. Schimansky-Geier. Effects of noise in excitable systems. *Physics Reports*, 392(6):321–424, 2004.

[417] W. Ebeling. Nonlinear Brownian motions – mean square displacement. *Cond. Mat. Phys.*, 7(3(39)):539–550, 2004.

[418] D. Ryter and U. Deker. Properties of the noise-induced "spurious" drift. I. *J. Math. Phys.*, 21:2662–2665, 1980.

[419] D. Ryter and U. Deker. Properties of the noise-induced ("spurious") drift. II. Simplifications of Langevin Equations. *J. Math. Phys.*, 21:2666–2669, 1980.

[420] W. Ebeling. Canonical non-equilibrium statistics and applications to Fermi-Bose systems. *Cond. Mat. Phys.*, 3(2):285–293, 2000.

[421] P. S. Epstein. On the Resistance Experienced by Spheres in their Motion through Gases. *Phys. Rev.*, 23:710, 1924.

[422] M. R. Hoare. The linear gas. *Adv. Chem. Phys.*, 20:135–214, 1971.

[423] M. R. Hoare, S. Raval, and M. Rahman. The Rayleigh Model: Singular Transport Theory in One Dimension. *Philos. Trans. R. Soc. Lond. A*, 305:383–440, 1982.

[424] S. A. Trigger. Fokker-Planck equation for Boltzmann-type and active particles: Transfer probability approach. *Phys. Rev. E*, 67:046403, 2003.

[425] R. Zwanzig. Nonlinear Generalized Langevin Equations. *J. Stat. Phys.*, 9(5):215–220, 1973.

[426] J. W. Rayleigh. *The Theory of Sound*, volume 1. Dover, New York, 2 edition, 1945.

[427] Yu. L. Klimontovich, A. S. Kovalev, and P. S. Landa. Natural fluctuations in lasers. *Physics-Uspekhi*, 15(1):95–113, 1972.

[428] S. Marksteiner, K. Ellinger, and P. Zoller. Anomalous diffusion and Lévy walks in optical lattices. *Phys. Rev. A*, 53:3409–3430, 1996.

[429] E. Lutz. Anomalous diffusion and Tsallis statistics in an optical lattice. *Phys. Rev. A*, 67:051402, 2003.

[430] S. A. Khrapak, A. V. Ivlev, G. E. Morfill, and S. K. Zhdanov. Scattering in the attractive Yukawa potential in the limit of strong interaction. *Phys. Rev. Lett.*, 90(22):225002, 2003.

[431] S. A. Khrapak, A. V. Ivlev, and G. E. Morfill. Momentum transfer in complex plasmas. *Phys. Rev. E*, 70:056405, 2004.

[432] S. A. Trigger, W. Ebeling, A. M. Ignatov, and I. M. Tkachenko. Fokker-Planck equation with velocity-dependent coefficients: Application to dusty plasmas and active particles. *Contrib. Plasma Phys.*, 43:377, 2003.

[433] A. G. Zagorodny, P. P. J. M. Schram, and S. A. Trigger. Stationary velocity and charge distributions of grains in dusty plasmas. *Phys. Rev. Lett.*, 84:3594–3597, 2000.

[434] A. S. Mikhailov. Selected topics in fluctuational kinetics of reactions. *Phys. Rep.*, 184(5-6):307–374, 1989.

[435] M. Schienbein and H. Gruler. Langevin Equation, Fokker-Planck Equation and Cell Migration. *Bull. Math. Biology*, 55:585–608, 1993.

[436] W. Ebeling, F. Schweitzer, and B. Tilch. Active Brownian particles with energy depots modelling animal mobility. *BioSystems*, 49:17–29, 1999.

[437] U. Erdmann, W. Ebeling, L. Schimansky-Geier, and F. Schweitzer. Brownian particles far from equilibrium. *Eur. Phys. J. B*, 15:105–113, 2000.

[438] F. Schweitzer, W. Ebeling, and B. Tilch. Complex Motion of Brownian Particles with Energy Depots. *Phys. Rev. Lett.*, 80(23):5044–5047, 1998.

[439] F. Schweitzer, W. Ebeling, and B. Tilch. Statistical mechanics of canonical-dissipative systems and applications to swarm dynamics. *Phys. Rev. E*, 64:021110, 2001.

[440] W. Ebeling, J. Dunkel, U. Erdmann, and S. A. Trigger. Klimontovich's contributions to the kinetic theory of nonlinear Brownian motion and new developments. *J. Phys.: Conf. Ser.*, 11:89–98, 2005.

[441] W. Ebeling, U. Erdmann, J. Dunkel, and M. Jenssen. Nonlinear dynamics and fluctuations of dissipative Toda chains. *J. Stat. Phys.*, 101(1/2):443–457, October 2000.

[442] W. Ebeling, P. S. Landa, and V. Ushakov. Self-oscillations in Toda ring chains with negative friction. *Phys. Rev. E*, 63:046601, 2001.

[443] V. A. Makarov, E. del Rio, W. Ebeling, and M. G. Velarde. Dissipative Toda-Rayleigh lattice and its oscillatory modes. *Phys. Rev. E*, 64:036601, 2001.

[444] J. Dunkel, W. Ebeling, U. Erdmann, and V. A. Makarov. Coherent motions and clusters in a dissipative Morse ring chain. *Int. J. Bif. & Chaos*, 12(11):2359–2377, 2002.

[445] S. Jespersen, R. Metzler, and H. C. Fogedby. Lévy flights in external force fields: Langevin and fractional Fokker-Planck equations and their solutions. *Phys. Rev. E*, 59(3):2736–2745, 1999.

[446] I. M. Sokolov. Lévy flights from a continuous-time process. *Phys. Rev. E*, 63(1):011104, 2000.

[447] A. Weron, K. Burnecki, S. Mercik, and K. Weron. Complete description of all self-similar models driven by Lévy stable noise. *Phys. Rev. E*, 71(1):016113, 2005.

[448] J. M. Sancho, A. M. Lacasta, K. Lindenberg, I. M. Sokolov, and A. H. Romero. Diffusion on a Solid Surface: Anomalous is Normal. *Phys. Rev. Lett.*, 92(25):250601, 2004.

[449] E. Heinsalu, M. Patriarca, I. Goychuk, G. Schmid, and P. Hänggi. Fractional Fokker-Planck dynamics: Numerical algorithm and simulations. *Phys. Rev. E*, 73(4):046133, 2006.

[450] I. Goychuk, E. Heinsalu, M. Patriarca, G. Schmid, and P. Hänggi. Current and universal scaling in anomalous transport. *Phys. Rev. E*, 73(2):020101, 2006.

[451] J. Blum, S. Bruns, D. Rademacher, A. Voss, B. Willenberg, and M. Krause. Measurement of the Translational and Rotational Brownian Motion of Individual Particles in a Rarefied Gas. *Phys. Rev. Lett.*, 97:230601, 2006.

[452] A. S. Kompaneets. The Establishment of Thermal Equilibrium between Quanta and Electrons. *Sov. Phys. JETP*, 4(5):730–737, 1957.

[453] P. A. M. Dirac. The conditions for statistical equilibrium between atoms, electrons and radiation. *Proc. Roy. Soc. London A*, 106:581–596, 1924.

[454] R. A. Sunyaev and Y. B. Zeldovich. The observations of relic radiation as a test of the nature of x-ray radiation from the clusters of galaxies. *Comm. Astrophys. Space Phys.*, 4:173, 1972.

[455] M. Birkinshaw, S. F. Gull, and H. Hardebeck. The Sunyaev-Zeldovich effect towards three clusters of galaxies. *Nature*, 309:34–35, 1984.

[456] M. Jones et al. An image of the Sunyaev-Zel'dovich effect. *Nature*, 365:320 – 323, 1993.

[457] P. R. Halmos. *Measure Theory*. The University Series in Higher Mathematics. D. Van Nostrand Comp., Inc., New York, 1951.

[458] W. Ochs. Basic properties of the generalized Boltzmann-Gibbs-Shannon entropy. *Rep. Math. Phys.*, 9(2):135–155, 1976.

[459] W. Ochs. A unique characterization of the generalized Boltzmann-Gibbs-Shannon entropy. *Rep. Math. Phys.*, 9(3):331–354, 1976.

[460] A. Wehrl. General properties of entropy. *Rev. Mod. Phys.*, 50(2):221–260, 1978.

[461] S. Kullback and R. A. Leibler. On information and sufficiency. *Ann. Math. Statistics*, 22:79–86, 1951.

[462] H. Goldstein. *Classical Mechanics*. Addison-Wesley series in physics. Addison-Wesley, Reading, 2 edition, 1980.

[463] A. Haar. Der Massbegriff in der Theorie der kontinuierlichen Gruppen. *Ann. Math.*, 34(1):147–169, 1933.

[464] B. J. Alder and T. E. Wainwright. Studies in Molecular Dynamics. General Method. *J. Chem. Phys.*, 31(2):459–466, 1959.

[465] J. Masoliver and J. Marro. Kinetics of a finite one-dimensional mixture of hard rods with different masses. *J. Stat. Physics*, 31(3):565–575, 1983.

[466] M. Abramowitz and I. A. Stegun, editors. *Handbook of Mathematical Functions*. Dover Publications, Inc., New York, 1972.

[467] Yu. M. Sinyukov. On the relativistic partition function of ideal gases. *Phys. Lett. B*, 127(6):443–447, 1983.

[468] A. S. Eddington. *The Mathematical Theory of Relativity*. University Press Cambridge, 1923.

[469] J. Dunkel, P. Hänggi, and S. Hilbert. Non-locality and thermodynamics in special relativity. *Submitted*, 2008.

[470] A. Fingerle. Relativistic fluctuation theorems: Theory and explicit examples. *Compt. Rend. Acad. Sci. Physique*, 8(5-6):696–713, 2007. arXiv:0705.3115v1.

[471] L. Zaninetti. X-ray profiles in symmetric and asymmetric supernova remnants. *Baltic Astron.*, 16:251–285, 2007.

[472] M. Rigotti. A relativistic stochastic process. Master's thesis, ETH Zürich, Institute for Theoretical Physics, 2005.

[473] P. Hänggi. Connection between deterministic and stochastic descriptions of nonlinear systems. *Helv. Phys. Acta*, 53:491–496, 1980.

[474] L. H. Ryder. *Quantum field theory*. Cambridge University Press, Cambridge, 2 edition, 1996.

[475] G. Roepstorff. *Path Integral Approach to Quantum Mechanics*. Springer, Berlin, 1996.

[476] R. Graaff and K. Rinzema. Practical improvements on photon diffusion theory: application to isotropic scattering. *Phys. Med. Biol.*, 46:3043–3050, 2001.

[477] D. C. Sahni, E. B. Dahl, and N. G. Sjöstrand. Diffusion coefficient for photon transport in turbid media. *Phys. Med. Biol.*, 48:3969–3976, 2003.

[478] J. Masoliver, J. M. Porrà, and George H. Weiss. Solutions of the telegrapher's equation in the presence of traps. *Phys. Rev. A*, 45(4):2222–2227, 1992.

[479] J. Masoliver, J. M. Porrà, and G. H. Weiss. Erratum: Solutions of the telegrapher's equation in the presence of traps. *Phys. Rev. A*, 46:3574, 1992.

[480] S. K. Foong and S. Kanno. Comment on "Solutions of the telegrapher's equation in the presence of traps". *Phys. Rev. A*, 46(8):5296–5297, 1992.

[481] M. Renardy. Asymptotic Solution of the Telegraph Equation (Mark A. Pinsky). *SIAM Review*, 35(2):306–307, 1993.

[482] A. Y. Dorogovtsev. Periodic in distribution solution for a telegraph equation. *J. Appl. Math. Stoch. Anal.*, 12(2):121–131, 1999.

[483] W. Thomson. On the Theory of the Electric Telegraph. *Proc. Roy. Soc. London, Series I*, 7:382–399, 1854-1855.

[484] M. O. Caceres. Non-Markovian Processes with Long-Range Correlations: Fractal Dimension Analysis. *Braz. J. Phys.*, 29(1):125–135, 1999.

[485] P. A. M. Dirac. Classical theory of radiating electrons. *Proc. Roy. Soc. London Ser. A*, 167(929):148–169, 1938.

[486] C. J. Eliezer. On the classical theory of particles. *Proc. Royal Soc. London*, 194:543–555, 1948.

[487] F. Rohrlich. Classical self-force. *Phys. Rev. D*, 60(8):084017, 1999.

[488] F. Rohrlich. The self-force and radiation reaction. *Amer. J. Phys.*, 68(12):1109–1112, 2000.

[489] H. Spohn. The Critical Manifold of the Lorentz-Dirac Equation. *Europhys. Lett.*, 50:287–292, 2000.

[490] G. Ares de Parga, R. MAres, and S. Dominguez. An unphysical result for the Landau-Lifshitz equation of motion for a charged particle. *Rev. Mexicana Fis.*, 52(2):139–142, 2006.

[491] P. R. Johnson and B. L. Hu. Stochastic Theory of Relativistic Particles Moving in a Quantum Field: I. Influence Functional and Langevin Equation. arXiv:quant-ph/0012137v4, 2001.

[492] P. R. Johnson and B. L. Hu. Worldline Influence Functional: Abraham-Lorentz-Dirac-Langevin Equation from QED. In P. Chen, editor, *Quantum Aspects of Beam Physics*, pages 95–104, Singapore, 2002. World Scientific. 18th Advanced IFCA Beam Dynamics Workshop, Capri 2000.

[493] P. R. Johnson and B. L. Hu. Stochastic Theory of Relativistic Particles Moving in a Quantum Field: Radiation Reaction, Vacuum Fluctuations, and the Abraham-Lorentz-Dirac-Langevin Equation. *Phys. Rev. D*, 65:065015, 2002.

[494] C. R. Galley, B. L. Hu, and S. Lin. Electromagnetic and gravitational self-force on a relativistic particle from quantum fields in curved space. *Phys. Rev. D*, 74(2):024017, 2006.

[495] E. Fermi. On the origin of the cosmic radiation. *Phys. Rev.*, 75(8):1169–1174, 1949.

[496] C. Jarzynski. Nonequilibrium Equality for Free Energy Differences. *Phys. Rev. Lett.*, 78(14):2690–2693, 1997.

[497] U. Seifert. Entropy Production along a Stochastic Trajectory and an Integral Fluctuation Theorem. *Phys. Rev. Lett.*, 95:040602, 2005.

[498] F. Douarche, S. Joubaud, N. B. Garnier, A. Petrosyan, and S. Ciliberto. Work fluctuation theorems for harmonic oscillators. *Phys. Rev. Lett.*, 97(14):140603, 2006.

[499] C. Jarzynski. Nonequilibrium Fluctuations of a Single Biomolecule. In *Controlled Nanoscale Motion*, volume 711 of *Lecture Notes in Physics*, pages 201–216. Springer, Berlin, Heidelberg, 2007.

[500] P. Talkner, E. Lutz, and P. Hänggi. Fluctuation theorems: Work is not an observable. *Phys. Rev. E*, 75(5):050102, 2007.

[501] B. L. Hu and E. Verdaguer. Stochastic gravity: Theory and applications. *Living Reviews in Relativity*, 7(3), 2004.

[502] H.-P. Breuer and F. Petruccine. A Lorentz covariant stochastic wave function dynamics for open systems. *Phys. Lett. A*, 242(4-5):205–210, 1998.

[503] H.-P. Breuer and F. Petruccine. Relativistic formulation of quantum state diffusion. *J. Phys. A: Math. Gen.*, 31(1):33–52, 1998.

[504] H.-P. Breuer and F. Petruccine. Reply to 'Relativistic formulation of quantum state diffusion?'. *J. Phys. A: Math. Gen.*, 31(47):9605–9612, 1998.

[505] L. Diosi. Relativistic formulation of quantum state diffusion? *J. Phys. A: Math. Gen.*, 31(47):9601–9603, 1998.

[506] H. Kleinert and S. V. Shabanov. Theory of Brownian motion of a massive particle in spaces with curvature and torsion. *J. Phys. A: Math. Gen.*, 31:7005–7009, 1998.

[507] C. Chevalier and F. Debbasch. Fluctuation-dissipation theorems in an expanding universe. *J. Math. Phys.*, 48(2):023304, 2007.

[508] F. Debbasch. A diffusion process in curved space-time. *J. Math. Phys.*, 45(7):2744–2760, 2004.

[509] M. Rigotti and F. Debbasch. An H-theorem for the general relativistic Ornstein-Uhlenbeck process. *J. Math. Phys.*, 46:103303, 2005.

Lebenslauf

Name:	Jörn Dunkel
Geburtsdatum:	12. Juni 1976
Geburtsort:	Berlin
Familienstand:	verheiratet mit Katharina Kraatz-Dunkel, geb. Kraatz

1996	Abitur, Alexander von Humboldt-Gymnasium, Berlin
1996-1997	Zivildienst, Lazarus-Hilfswerk Berlin
1999	Vordiplom Physik, Humboldt-Universität zu Berlin
2000	Vordiplom Mathematik, Humboldt-Universität zu Berlin
2000-2001	Visiting Student, Exeter College, University of Oxford
2004	Physikdiplom, Humboldt-Universität zu Berlin
2005	Mathematikdiplom, Humboldt-Universität zu Berlin
1998-2002	Studentische Hilfskraft, Humboldt-Universität zu Berlin
2005-2006	Wissenschaftlicher Mitarbeiter, Max-Planck-Institut für Astrophysik (Garching)
2006-2008	Wissenschaftlicher Mitarbeiter, Universität Augsburg

Danksagung

Ich möchte mich bei allen Kollegen und Freunden bedanken, die im Laufe der vergangen Jahre durch Vorlesungen, Bemerkungen und Kommentare direkt oder indirekt zur der vorliegenden Dissertation beigetragen haben. Besonders großer Dank gilt dabei meinem Betreuer Peter Hänggi, ohne dessen Anregungen, Unterstützung und Enthusiasmus diese Arbeit nicht möglich gewesen wäre. Ein weiteres herzliches Dankeschön richtet sich an Peter Talkner und Gert Ingold für ihre bewundernswerte Geduld beim Beantworten meiner Fragen. Also, many thanks to David and Jesus (und Stefan) for the fruitful collaboration.

Zurückblickend bin ich sehr glücklich darüber, daß ich während der vergangenen drei Jahre in einer so angenehmen, freundlichen und hilfsbereiten Umgebung arbeiten und lernen durfte. Dafür möchte ich allen ehemaligen und gegenwärtigen Mitarbeitern, Postdocs, Doktoranden und Zimmerkollegen aus `theo1` nocheinmal ganz herzlich danken. Ihr alle habt dafür gesorgt, daß ich das tägliche Pendeln zwischen Augsburg und München gerne in Kauf genommen habe!